$$f_X(x) = c_k \cdot \sin^k(x),\ x \in (0,\pi),\ k = 1,2,3\ldots\#\text{columns} - 1, \text{ and } c_k = \frac{\Gamma(k/2+1)}{\sqrt{\pi}\,\Gamma(k/2+1/2)}$$

$$F_X(x;k) \sim \frac{1}{2} - \left(\frac{1}{2}\right)\cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x < \frac{\pi}{2}, \qquad \sim \frac{1}{2} + \left(\frac{1}{2}\right)\cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x \geq \frac{\pi}{2}$$

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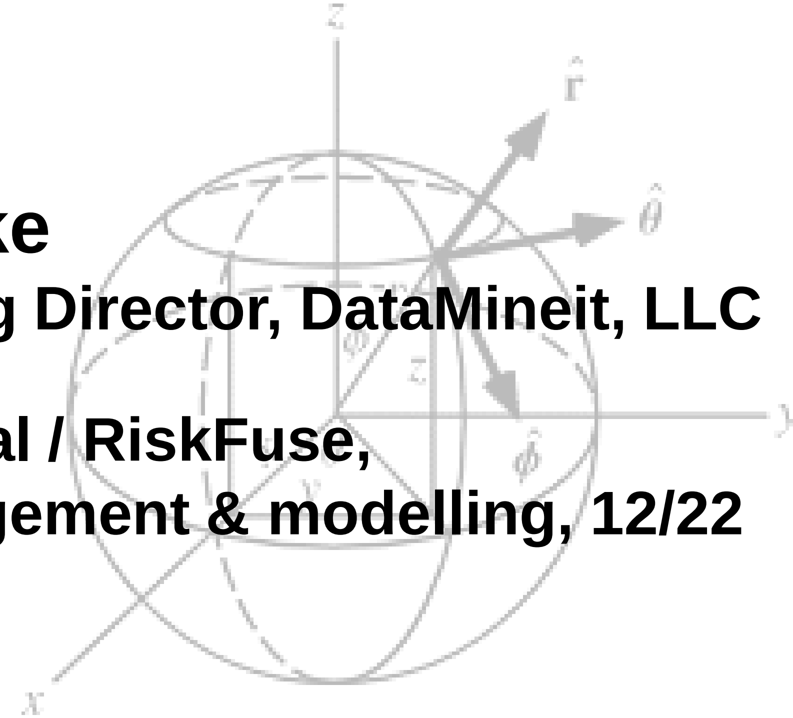
$$F^{-1}(p;k) = \arccos\left(\sqrt{F^{-1}_{Beta}\left(1 - 2p; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p < 0.5;$$



JD Opdyke

Chief Analytics Officer, Sr. Managing Director, DataMineit, LLC

RiskMinds International / RiskFuse,
AI & ML applications for risk management & modelling, 12/22

$$= \pi - \arccos\left(\sqrt{F^{-1}_{Beta}\left(1 - 2[1-p]; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p \geq 0.5$$

Acknowledgments & Disclaimer

The views presented herein are solely those of the author and do not necessarily reflect the views of Allstate, DataMineit, LLC, or any of their subsidiaries or legal entities.

JD Opdyke, the sole author of the work contained herein, presents this paper in the public domain with the permission of, and gratitude to, Mark Prindiville, PhD, CRO – Allstate. The author also would like to thank Dan Kirsner, PhD and Ming Dai, PhD for reviewing an early draft of this presentation. Any errors are my own.

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Book Chapter Abstract

Responsible use of any portfolio model that incorporates correlation structure requires knowledge of its sampling distribution. This is especially true of models used in stress testing, or ones requiring the specification of particular scenarios with particular correlation values (e.g. in views-based portfolio analyses a la Black-Litterman (1991) and its variants), because there is no other way to reliably associate those values with probabilities. Put differently, the correlation matrix is a model parameter like any other, and a model's results cannot be fully understood or relied upon if the behavior of its parameters is not well defined. This makes answering the following question of central importance: given an estimated (product-moment) correlation matrix and an assumed or well-estimated data generating mechanism, what is the sampling distribution of that matrix? And how does that finite sample density relate to the densities of each of its pairwise correlation cells?

We provide the solution to this question for any correlation matrix under the most general data conditions possible, requiring only the existence of the mean and variance for each marginal distribution in the portfolio and the positive definiteness of the correlation matrix. This solution explicitly connects the densities associated with each of the correlation cells to that of the entire matrix, making the latter a function of the former. One only needs to specify the cumulative distribution function (cdf) value associated with each of the cells to retrieve the corresponding, unique correlation matrix (as well as its overall probability of occurrence conditional on the estimated matrix). And in reverse, given a valid (positive definite) correlation matrix, the density provides the corresponding, unique matrix of cdf values, as well as its overall conditional probability of occurrence.

Associating matrices with their 'cdf matrices' can be defined in terms of shifts from the estimated correlation matrix, providing a very convenient and flexible way to specify scenarios at the most granular level – that of every pairwise relationship between factors/variables, as opposed to merely at the level of the factors/variables. This all is accomplished within a geometric framework wherein the sampling mechanism automatically enforces positive definiteness, which not only allows for efficient enumeration of the sampling space, but also and more importantly, much more robust inference of the entire matrix compared to other more limited (spectral) approaches.

Finally, unlike any other framework, the geometric approach to this problem also scales on itself: it can be applied to any submatrix of the given correlation matrix, enabling fully flexible scenario specification wherein some cells remain completely untouched, while others are appropriately affected by the scenario. This is a realistic, common need in many investment and risk settings.

The foundations of the geometric approach have long been established within the relevant statistical and finance literatures, but its pieces have not previously been combined in such a way as to solve this problem under general conditions. All results are validated by well-established results from the Random Matrix Theory literature, even as the geometric approach is more robust, empirically and structurally, than those relying on spectral distributions.

Lastly, the solution is scalable, having been readily implemented on a commodity laptop on matrices 100x100 and larger.

Contents

I. Objective

- Responsible use of any (portfolio) model that incorporates Pearson's correlation matrix* requires knowledge of its sampling distribution. To date, an easily implemented, robust, and fully flexible solution valid under general conditions has remained elusive, even as this is often the most impactful parameter in such models. Knowledge of its finite sample density is especially important for portfolio models used in stress testing, or requiring the specification of particular scenarios with particular correlation values (e.g. views-based portfolio analyses a la Black-Litterman (1991) and its variants).
- **Objective**: Develop a method to obtain the finite sample density of the correlation matrix, and its inverse ('quantile function'), under the most general conditions possible, requiring only the existence of the mean, the variance, and positive definiteness. Important characteristics:
  - **Fully Flexible**: make this density
    i. a direct function of the densities associated with each of the correlation matrix cells, and
    ii. valid and applicable to **any subset of cells in the matrix** while holding the rest constant

    Satisfying i. and ii. allows for fully flexible stress testing and scenario specification, with the added benefit that specific scenarios for the correlation matrix now can be defined probabilistically.
  - **Scalable**: reasonably fast for reasonably high dimensions (e.g. 100x100), with an implementation that does not change or become unwieldy in higher dimensions.
  - **Robust inference**, even when the matrix approaches singularity (i.e. non-positive definiteness).
  - **Accurate**: does not rely on approximations that can be inaccurate under conditions that are not uncommon in financial portfolios (e.g. Fisher Z transformation under near singularity).
- Requirements: The above is conditional on 1. a specified or well estimated correlation matrix (this is a very rich literature) and 2. a specified or well estimated data generating mechanism.

* Unless otherwise noted, "correlation" herein refers to Pearson's product moment correlation (see Pearson, 1895).

II. Definitions

- Pearson's product moment correlation is defined as ρ; its sample analog r uses sample moments:

$$\rho = \frac{\sum_{i=1}^{N}\left(X_i - E(X)\right)\left(Y_i - E(Y)\right)}{\sqrt{\sum_{i=1}^{N}\left(X_i - E(X)\right)^2}\sqrt{\sum_{i=1}^{N}\left(X_i - E(X)\right)^2}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \qquad r = \frac{\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{i=1}^{n} X\right)\left(Y_i - \frac{1}{n}\sum_{i=1}^{n} Y\right)}{\sqrt{\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{i=1}^{n} X\right)^2}\sqrt{\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{i=1}^{n} Y\right)^2}} = \frac{Cov(X,Y)}{s_X s_Y}$$

- The corresponding correlation matrix R is the matrix of all pairwise correlations, with the following characteristics:
  - Symmetry: $r_{i,j} = r_{j,i}$
  - Unit diagonal entries: $r_{i=j} = 1$
  - *Bounded non-diagonal entries: $-1 \le r_{i,j} \le 1$
  - The matrix is positive definite, i.e. all eigenvalues $\lambda_i > 0$

$$R = \begin{bmatrix} 1 & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & 1 & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & 1 & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & 1 \end{bmatrix}$$

  - For completeness, we define eigenvalues below:
    If there exists a nonzero vector v such that $Rv = \lambda v$ then λ is an eigenvalue of R and v is its corresponding eigenvector. λ and v can be obtained by solving

$\det(\lambda I - R) = 0$, then $\det(\lambda I - R)v = 0$, where I is the identity matrix and det is the determinant

The eigenvalue can be thought of as the magnitude of the (portfolio) variance in the direction of the eigenvector.

* Of course, these can be tighter under specific circumstances, such as for equicorrelation matrices where $-1/(n-1) \le \rho \le 1$.

III.1 Literature: Identity Matrix & Gaussian Data

- For the identity matrix (correlations all equal zero) under Gaussian data and d≥2, Gupta & Nagar (2000) provide the pdf

$$f(R)=\frac{\left[\Gamma\left(\left[n-1\right]/2\right)\right]^2 |R|^{(n-p-2)/2}}{\Gamma_p\left(\left[n-1\right]/2\right)}, \quad \text{where } |\;| \text{ is the determinant function, } -1\le r_{i,j}=r_{j,i}\le 1,\ r_{i,i}=1,\ 1\le i,j\le p \text{ and } \Gamma_p\left(\left[n-1\right]/2\right)=\pi^{\left[p(p-1)/4\right]}\prod_{i=1}^{p}\Gamma\left(\left[n-i\right]/2\right)$$

  where *R* is the empirical correlation matrix, *p* is its dimension (#rows/cols), *n* is the sample size, and Γ is the gamma function. However, we focus below on methods for efficiently sampling this density.

  - The c-vines and onion methods (Lewandowski et al., 2009)
  - The restricted Wishart distribution approach of Wang et al. (2018).
  - The direct formulation method of Madar (2015)
  - The Cholesky-Metropolis method of Cordoba et al. (2018)
  - The polar angles distribution of Makalic and Schmidt (2018) combined with the polar representation of Pinheiro & Bates (1996), Rebonato & Jaeckel (2000), & Rapisarda et al. (2007).

- The geometric interpretation provided by the polar angles approach is based on the fact that the cosine of the angle between two mean-centered vectors X, Y is equal to Pearson's correlation:

$$\cos(\theta)=\frac{\langle \mathbf{X},\mathbf{Y}\rangle}{\|\mathbf{X}\|\|\mathbf{Y}\|}=\frac{\sum_{i=1}^{N}\left(X_i-E(X)\right)\left(Y_i-E(Y)\right)}{\sqrt{\sum_{i=1}^{N}\left(X_i-E(X)\right)^2}\sqrt{\sum_{i=1}^{N}\left(X_i-E(X)\right)^2}}=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=\rho, \quad \text{with } 0\le\theta\le\pi$$

- The framework of our proposed method relies on this geometric interpretation, so it is worth examining one sampling implementation based on it here.

III.1 Literature: Identity Matrix & Gaussian Data

- Of the sampling methods listed above, Makalic and Schmidt (2018) arguably is the most simple and among the fastest (see Cordoba et al., 2019).
- The polar angles approach to sampling the (Gaussian data) identity matrix is an efficient recognition of the fact that merely perturbing non-diagonal correlation values uniformly, between -1 and 1, will generate mostly non-positive definite matrices. In fact, Bohn & Hornik (2014) and Pourahmadi & Wang (2015) show that the ratio of valid correlation matrices to all matrices generated this way that LOOK like correlation matrices is

$$\Pr\left(rand\,\text{"}R\text{"} \sim PosDef\right) = X = \frac{\prod_{j=1}^{p-1}\left[\sqrt{\pi}\,\Gamma\left(\frac{j+1}{2}\right)\right]^{j}}{2^{p(p-1)/2}} < \prod_{j=1}^{p-1}\left[\frac{\sqrt{\pi}}{2}\right]^{j} = \left[\frac{\sqrt{\pi}}{2}\right]^{p(p-1)/2} ;\lim_{p\to\infty}\left[X\right] = 0$$

- For even relatively small matrices of dimension p=25, the odds of successfully randomly generating a single valid correlation matrix are less than 2 in ten quadrillion: hence, for the sake of computational efficiency, we need to ONLY generate valid, positive definite matrices by constraining our sampling to those matrices on the hyper hemisphere, as described below.

III.1 Literature: Identity Matrix & Gaussian Data

- The Cholesky factorization of a correlation matrix (see below) has rows whose squares sum to 1.0, so it is commonly used as a convenient way to ensure that samples remain on the unit hypersphere (or technically in this case, the unit hyper-hemisphere of dimension *p*).

- Pourahmadhi & Wang (2015) and others show that the uniform distribution of positive definite matrices on the p-dimensional hemisphere is proportional to the determinant of the Jacobian, which is defined in terms of the Cholesky factorization as shown below (see also Cordoba et al., 2018)

$$\det\left[J\left(U\right)\right]=2^{p}\prod_{i=1}^{p-1}u_{ii}^{i}\quad \text{where } U \text{ is the Cholesky factorization of correlation matrix } R=UU^{t}$$

- So Makalic and Schmidt (2018) and others (see Pourahmadi & Wang, 2015) recognized that sampling polar angles based on pdf

$$f_X\left(x\right)=c_k\cdot\sin^k\left(x\right),\ x\in\left(0,\pi\right),\ k=1,2,3\ldots\#\text{columns}-1,\text{ and } c_k=\frac{\Gamma\left(k/2+1\right)}{\sqrt{\pi}\,\Gamma\left(k/2+1/2\right)}$$

satisfies this constraint. Although not mentioned in Makalic and Schmidt (2018), importantly note that k = #columns – column# (so for the first column of a p=10x10 matrix, k=9; for the second column, k=8, etc.). So the spread of the angles distributions is a function of the column number (as shown in graphical results in the Appendices below).

III.1 Literature: Identity Matrix & Gaussian Data

- For completeness, we include below the definition of the Cholesky factor and corresponding formulae:
- A correlation matrix *R* will be real, symmetric positive-definite, so the unique matrix *B* that satisfies

$$R = BB^T$$

where *B* is a lower triangular matrix (with real and positive diagonal entries), and B^T is its transpose, is the Cholesky factorization of *R*. Formulaically, *B*'s entries are as follows:

$$B_{j,j} = (\pm)\sqrt{R_{j,j} - \sum_{k=1}^{j-1} B_{j,k}^2} \qquad B_{i,j} = \frac{1}{B_{j,j}}\left(R_{i,j} - \sum_{k=1}^{j-1} B_{i,k}B_{j,k}\right) \text{ for } i > j$$

- The Cholesky factorization can be thought of as the matrix analog to the square root of a scalar.

III.1 Literature: Identity Matrix & Gaussian Data

- For sampling from parametric distributions, ideally the cdf can be inverted analytically, and then inverse probability sampling can be used. But in this case, Maklic and Schmidt (2018) state:

  "Generating random numbers from this distribution is not straightforward as the corresponding cumulative density [sic] function, although available in closed form, is defined recursively and requires O(k) operations to evaluate. The nature of the cumulative density [sic] function makes any procedure based on inverse transform sampling computationally inefficient, especially for large k."

- However, we shall see later that, in fact, Opdyke (2019) derives the analytical cdf of this distribution, as well as its analytical inverse, thus enabling the use of the inverse probability transform for more efficient sampling here, especially for large k (i.e. for large dimensional matrices).

- Maklic & Schmidt's (2018) approach was rejection sampling with a scaled beta distribution envelop:

$$\text{1. Generate X} \sim \text{Beta}\left(k+1,k+1\right) \sim \frac{x^k\left(\pi-x\right)^k}{B\left(k+1,k+1\right)\pi^{(2k+1)}},\ x\in\left(0,\pi\right),\ k\geq 1, B\left(q,r\right)=\frac{\Gamma\left(q\right)\Gamma\left(r\right)}{\Gamma\left(q+r\right)}=\text{Euler's beta function}$$

$$\text{2. Generate U} \sim \text{Uniform}\left(0,1\right)$$

$$\text{3. Accept X if } \frac{\ln\left(U\right)}{k}\leq \ln\left(\frac{\pi^2\sin\left(X\right)}{4X\left(\pi-X\right)}\right) \qquad \text{4. Otherwise go to 1.}$$

- The algorithm's maximum expected iterations per sample is $\pi/\left(2\sqrt{2}\right)=1.11$, making it roughly 10% less efficient than an analytical solution.

III.1 Literature: Identity Matrix & Gaussian Data

- The more general and important point here is that **we have a direct relationship between the angles between two data vectors, and their corresponding correlations**. Translation between the two is straightforward and non-directional. This will be explored in depth later, but in summary:

  1. estimate the correlation matrix
  2. obtain the Cholesky factorization of the correlation matrix
  3. Use inverse trigonometric functions on 2. to obtain corresponding spherical angles

  And in reverse:

  3. Start with a matrix of spherical angles
  2. apply trigonometric functions to obtain the Cholesky factorization
  1. multiply 2. by its transpose to obtain the corresponding correlation matrix

  see Rebonato & Jaeckel (2000) and Rapisarda et al. (2007) (note a typo in the formula in Pourahmadi & Wang (2015) for the first 3 steps)

- Note that **this relationship is true generally, not only for the identity matrix under Gaussian data.**

- Note also the inverse relationship between angles and correlations: correlations decrease monotonically in their corresponding angles, i.e. correlations increase as angles decrease to zero, and decrease as angles increase to π (see Zhang et al. (2015) and Lu et al. (2019)). The range from 0 to π rather than 0 to 2π is why this is the p-dimensional hyper hemisphere rather than the hypersphere.

III.1 Literature: Identity Matrix & Gaussian Data

So for R, a p x p correlation matrix:

$$R = \begin{bmatrix} 1 & r_{1,2} & r_{1,3} & \cdots & r_{1,p} \\ r_{2,1} & 1 & r_{2,3} & \cdots & r_{2,p} \\ r_{3,1} & r_{3,2} & 1 & \cdots & r_{3,p} \\ r_{4,1} & r_{4,2} & r_{4,3} & \cdots & r_{4,p} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ r_{p,1} & r_{p,2} & r_{p,3} & \cdots & 1 \end{bmatrix}$$

$R = BB^t$ where B is the Cholesky factor of R and

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \cos(\theta_{2,1}) & \sin(\theta_{2,1}) & 0 & \cdots & 0 \\ \cos(\theta_{3,1}) & \cos(\theta_{3,2})\sin(\theta_{3,1}) & \sin(\theta_{3,2})\sin(\theta_{3,1}) & \cdots & 0 \\ \cos(\theta_{4,1}) & \cos(\theta_{4,2})\sin(\theta_{4,1}) & \cos(\theta_{4,3})\sin(\theta_{4,2})\sin(\theta_{4,1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cos(\theta_{p,1}) & \cos(\theta_{p,2})\sin(\theta_{p,1}) & \cos(\theta_{p,3})\sin(\theta_{p,2})\sin(\theta_{p,1}) & \cdots & \prod_{k=1}^{n-1}\sin(\theta_{p,k}) \end{bmatrix}$$

for $i > j$ angles $\theta_{i,j} \in (0,\pi)$.

So from $R = BB^t$, for an individual pairwise correlation we have

$$r_{i,j} = \cos(\theta_{i,1})\cos(\theta_{j,1}) + \prod_{k=2}^{i-1}\cos(\theta_{i,k})\cos(\theta_{j,k})\prod_{l=1}^{k-1}\sin(\theta_{i,l})\sin(\theta_{j,l}) + \cos(\theta_{j,i})\prod_{l=1}^{i-1}\sin(\theta_{i,l})\sin(\theta_{j,l}) \quad \text{for } 1 \le i < j \le n$$

And recursively, in the other direction, for an individual angle $\theta_{i,j}$ we have:

For $i > 1$: $\theta_{i,1} = \arccos(b_{i,1})$ for j=1; and $\theta_{i,j} = \arccos\left(b_{i,j} \Big/ \prod_{k=1}^{j-1}\sin(\theta_{i,k}) \right)$ for $j > 1$

III.1 Literature: Identity Matrix & Gaussian Data

Correlations to Angles

```
* INPUT rand_R is a valid correlation matrix;

cholfact = T(root(rand_R, "NoError"));

rand_corr_angles = J(nrows,nrows,0);
  do j=1 to nrows;
    do i=j to nrows;
      if i=j then rand_corr_angles[i,j]=.;
      else do;
        cumprod_sin = 1;
        if j=1 then rand_corr_angles[i,j]=arcos(cholfact[i,j]);
        else do;
          do kk=1 to (j-1);
            cumprod_sin = cumprod_sin*sin(rand_corr_angles[i,kk]);
          end;
          rand_corr_angles[i,j]=arcos(cholfact[i,j]/cumprod_sin);
        end;
      end;
    end;
  end;

* OUTPUT rand_corr_angles is the corresponding matrix of angles;
```

SAS/IML code (v9.4)

Angles to Correlations

```
* INPUT rand_angles is a valid matrix of correlation angles;

Bs=J(nrows, nrows, 0);
do j=1 to nrows;
  do i=j to nrows;
    if j>1 then do;
      if i>j then do;
        sinprod=1;
        do gg=1 to (j-1);
          sinprod = sinprod*sin(rand_angles[i,gg]);
        end;
        Bs[i,j]=cos(rand_angles[i,j])*sinprod;
      end;
      else do;
        sinprod=1;
        do gg=1 to (i-1);
          sinprod = sinprod*sin(rand_angles[i,gg]);
        end;
        Bs[i,j]=sinprod;
      end;
    end;
    else do;
      if i>1 then Bs[i,j]=cos(rand_angles[i,j]);
      else Bs[i,j]=1;
    end;
  end;
end;
rand_R = Bs*T(Bs);

* OUTPUT rand_R is the corresponding correlation matrix;
```

III.1 Literature: Identity Matrix & Gaussian Data

- These geometric relationships all are well established in the statistical sampling literature, and are key to the method presented later herein.
- Limitations: Of course, all the methods listed above for sampling the correlation matrix, including that of Makalic and Schmidt (2018), have two major limitations: they only are valid when the true correlation matrix is the identity matrix, and they only are valid when the underlying data is Gaussian.
- While the identity matrix is fundamental and useful for testing one specific hypothesis, it arguably is the least interesting or useful for general inference since for any given instance in practice, the occurrence of zero's in all non-diagonal cells is highly improbable. Also, the requirement that data is strictly multivariate Gaussian is quite restrictive.

- **<u>Case of d=2</u>:**
  Taraldsen (2021) derived the exact sampling/empirical distribution of Pearson's correlation under Gaussian data for the pairwise case, i.e. for d=2, as below:

$$\pi(\rho\,|\,r)=\frac{\left(1-r^2\right)^{(v-1)/2}\cdot\left(1-\rho^2\right)^{(v-2)/2}\cdot\left(1-r\rho\right)^{(1-2v)/2}}{\sqrt{2}B\left(v+1/2,1/2\right)}\cdot {}_2F_1\left(\frac{3}{2},-\frac{1}{2};v+\frac{1}{2};\frac{1+r\rho}{2}\right)\ \text{where}$$

$B(X,Y)=\left[\Gamma(X)\Gamma(Y)\right]/\Gamma(X+Y)$ the Beta function, $v=n-1>1,$ and F is the Gaussian hypergeometric function where

$${}_2F_1[a,b;c;z]=\sum_{n}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}\ \text{ where }\ (h)_n=h(h+1)(h+2)\cdots(h+n-1),\ n\geq 1,\ (h)_0=1$$

- Note that **Taraldsen (2021) shows that the approximate density of the pairwise correlation using Fisher's Z-transformation loses accuracy** as $\rho\rightarrow 1$, especially for smaller samples.

- **<u>Case of d>2</u>:** Pham-Gia & Choulakian (2014) provide

$$f(R)=\frac{\left[\Gamma\left(\left[n-1\right]/2\right)\right]^2\exp\left\{-\sum_{i<j}\frac{\lambda_{i,j}s_{i,j}}{\sqrt{\sigma_{i,i}\sigma_{j,j}}}\right\}}{\Gamma_p\left(\left[n-1\right]/2\right)\left[|\Lambda|\prod_{i=1}^{p}\lambda_{i,i}\right]^{(\left[n-1\right]/p)}}|R|^{(n-p-2)/2}$$

with sample covariance $\{s_{i,j}\}$, $1\leq i<j\leq p$, $\Gamma_p\left(\left[n-1\right]/2\right)=\pi^{\left[p(p-1)/4\right]}\prod_{i=1}^{p}\Gamma\left(\left[n-i\right]/2\right)$, known variance $\{\sigma_{i,i}\}$, $|\ |$ is the determinant function, Λ is the true correlation matrix, and $\lambda_{i,i}$ the diagonals of Λ^{-1}

- <u>Limitations</u>: Application of Pham-Gia & Choulakian (2014) above and their equivalent expressions requires a priori knowledge of true (not estimated) variances. It also arguably is quite cumbersome to implement. Finally, it remains valid only under the fairly restrictive case of multivariate Gaussian data.

III.3 Literature: General Conditions

- Archakov & Hansen (2021) introduce a parameterization of the correlation matrix that maps uniquely, one-to-one, to the positive definite space, thus providing a density for inference. It is based on the Fisher Z transformation, remains invariant to reorderings of the variables, and is accompanied by an algorithm that provides the inverse mapping from the parameterization to the correlation matrix.

- Limitations: The authors state, "This makes the transformation potentially useful for … inference. These attributes tend to deteriorate as *C* approaches singularity. This is not unexpected, because it is also true for the Fisher transformation when the correlation is close to ±1."

- As previously noted, Taraldsen (2021) shows that the approximate density of the pairwise correlation using Fisher's Z-transformation loses accuracy as $\rho \to 1$, especially for smaller samples. This is consistent with the authors' comments here, but they state this may only be material under extreme conditions. All else equal, having a method that avoided this non-robustness issue altogether would be preferrable.

- The method only provides the density of the entire correlation matrix: it does not appear to be able to modify correlation matrices, cell-by-cell, probabilistically, based on their individual densities. Again, this may not be an objective of the method, but all else equal, it is a very useful feature for stress testing and scenario analysis. The method we present herein – NAbC – has this capability.

- Finally, it is unclear whether the method can be applied successfully to submatrices of the correlation matrix (while holding the rest of the matrix constant). This is relevant as many reasonable scenario specifications would require this. The NAbC method we present herein has this capability also.

III.3 Literature: General Conditions

- Lan et al. (2020) take a fully Bayesian approach to this problem for both covariance and correlation matrices. Similar to the NAbC method we present herein, they use the Cholesky factorization to maintain positive definiteness, and by defining distributions on spheres as we do, utilize a large class of flexible prior distributions.

- Limitations: This approach is very comprehensive, involved, and includes estimation, which our method does not. However, it appears that for modeling correlation matrices specifically, their approach has some limitations. The authors state: "The priors for correlation matrix specified through the sphere-product representation are in general dependent among component variables. For example, the method we use to induce uncorrelated prior between y_i and $y_j \left(i < j\right)$ by setting $l_{jk} \approx 0$ for $k \leq i$ has a direct consequence that $\text{Cor}\left(y_i', y_j\right) \approx 0$ for $i' \leq i$. In another word, more informative priors (part of the components are correlated) may require careful ordering in $\{y_i\}$. To avoid this issue, one might consider the inverse of covariance (precision) matrices instead. This leads to modeling the *conditional* dependence, or *Markov network* … Our proposed methodology applies directly to (dynamic) precision matrices/processes, which will be our future direction."

- The method we develop herein (NAbC) for correlation matrix inference and stress testing/scenario specification can be applied successfully to ANY submatrix, while holding the rest of the matrix constant, without any unintended 'dependencies' and all while automatically enforcing positive definiteness. In fact, our approach fixes the 'unintended dependencies' problem that other researchers note, yet fail to control (this is discussed later). This perfect control, at the correlation cell level and ANY combination thereof, is very powerful, useful, and important as fully flexible scenario specification requires nothing short of this.

III.3 Literature: General Conditions

- Ghosh et al. (2020) also take a fully Bayesian approach to this problem, and just like our approach, they reparameterize Cholesky factors in terms of hyperspherical coordinates where the angles vary freely in the range $[0, \pi)$. Their focus is on estimation, although as a Bayesian approach it is comprehensive.

- Limitations: This approach is involved, and includes estimation, which our method does not. However, its use is restricted to parametric priors, which may limit its implementation under complex real world conditions (some of which are empirically implemented herein). In contrast, NAbC makes use of fully flexible nonparametric kernels that fit ANY angles distribution resulting from ANY data generating mechanism (with finite first and second moments). Also, Ghosh's et al. (2020) approach does not appear to have the capability of modeling any submatrix while leaving select cells of the correlation matrix 'untouched.' This perfect control, at the correlation cell level and ANY combination thereof, is very powerful, useful, and important as fully flexible scenario specification requires nothing short of this. This is one of the advantages of the NAbC method we develop herein and present below.

III.3 Literature: General Conditions

- Papenbrock et al. (2021) develop a novel approach to simulating correlation matrices for financial markets using evolutionary algorithms. It allows for the flexible yet robust incorporation of many observed features of real world financial correlation matrices. The algorithm scales well and can be used for backtesting, pricing, and hedging correlation-dependent investment strategies and financial products.

- Limitations: This approach does not appear to automatically enforce positive definiteness in its simulations. Enforcing positive definiteness ex post is less efficient, and always introduces systematic changes in the resulting matrices; here it could very well materially alter the very characteristics the algorithm is attempting to embed in the synthetic data. Because the method we propose below (NAbC) automatically enforces positive definiteness while also providing perfect control at the correlation cell level to specify ANY combination of cells for perturbation, it may well be the perfect partner for the evolutionary approach. It could be applied as a very flexible filter for invalidly generated correlation matrices, probabilistically ensuring that the correlation density generated is not systematically biased or skewed in unintended ways. It also could asses the degree to which synthetically generated ‘clusters’ are outlying, probabilistically.

IV.1 New Results: Identity Matrix & Gaussian Data

- Opdyke (2019) derived the analytic cdf for the pdf used by Makalic and Schmidt (2018):

$$f_X(x) = c_k \cdot \sin^k(x),\ x \in (0,\pi),\ k = 1,2,3\ldots\#\text{columns} - 1,\ \text{and } c_k = \frac{\Gamma(k/2+1)}{\sqrt{\pi}\,\Gamma(k/2+1/2)}$$ as…

$$F_X(x;k) \sim \frac{1}{2} - c_k \cdot \cos(x) \cdot {}_2F_1\left[\frac{1}{2}, \frac{1-k}{2}; \frac{3}{2}; \cos^2(x)\right] \text{ for } x < \frac{\pi}{2},$$

$$\sim \frac{1}{2} + c_k \cdot \cos(x) \cdot {}_2F_1\left[\frac{1}{2}, \frac{1-k}{2}; \frac{3}{2}; \cos^2(x)\right] \text{ for } x \geq \frac{\pi}{2}$$

where the Gaussian hypergeometric function $_2F_1[a,b;c;r] = \sum_n^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{r^n}{n!}$

where $(h)_n = h(h+1)(h+2)\cdots(h+n-1),\ n \geq 1,\ (h)_0 = 1$

But this can be simplified further using two established (if not obscure) identities.

For $c = a+1$ and $|r| < 1$ simultaneously, we have $_2F_1[a,b;c;r] = B(r;a,1-b)(a/r^a)$

where $B(r;a,b) = \int_0^r u^{a-1}(1-u)^{b-1}\,du$ = the incomplete beta function

In addition, we have

$$F_{Beta}(r;a,b) = B(r;a,b)/B(a,b) \text{ where } B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \text{the complete beta function, so}$$

$$B(r;a,b) = F_{Beta}(r;a,b) \cdot B(a,b)$$

* Note that the (Gaussian) hypergeometric function is not uncommon in this setting, making an appearance in derivations of the distribution of individual correlations (see Muirhead, 1982, and Taraldsen, 2021), moments of the spectral distribution under some conditions (see Adams et al. 2018, and https://reference.wolfram.com/language/ref/MarchenkoPasturDistribution.html), and the definition of positive definite functions (Franca & Menegatto, 2022).

IV.1 New Results: Identity Matrix & Gaussian Data

- Taken together we have:

$$F_X(x;k) \sim \frac{1}{2} - c_k \cdot \cos(x) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(\frac{2+k}{2}\right)} \cdot \left([1/2]/\sqrt{\cos^2(x)}\right) \text{ for } x < \frac{\pi}{2},$$

$$F_X(x;k) \sim \frac{1}{2} + c_k \cdot \cos(x) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(\frac{2+k}{2}\right)} \cdot \left([1/2]/\sqrt{\cos^2(x)}\right) \text{ for } x \geq \frac{\pi}{2}$$

- Recognizing that the complete Beta function is the inverse of the normalization factor of c(k) for these values, their product equals 1 and cancels, as do the two cosine terms, and we have:

$$F_X(x;k) \sim \frac{1}{2} - \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x < \frac{\pi}{2},$$

$$\sim \frac{1}{2} + \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x \geq \frac{\pi}{2}$$

- The cdf of the well-known Beta distribution is so straightforward that it readily can be used in a spreadsheet. And now, we can even obtain an analytic* quantile function of the angle distribution:

* Note that we use the term 'analytic' as opposed to 'closed-form' because we are unaware of a closed-form algorithm for the inverse cdf of the beta distribution (see Sharma and Chakrabarty, 2017, and Askitis, 2017). However, for all practical purposes this is essentially a semantic distinction since this quantile function is hard-coded into all major statistical / econometric / mathematical programming languages.

IV.1 New Results: Identity Matrix & Gaussian Data

Let $p = \Pr(x \geq X)$. Then for $x < \frac{\pi}{2}$,

$$p = \frac{1}{2} - \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right]$$

$$-2p = -1 + F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right]$$

$$1-2p = F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right]$$

$$F_{Beta}^{-1}\left(1-2p; \frac{1}{2}, \frac{1+k}{2}\right) = \cos^2(x)$$

$$\sqrt{F_{Beta}^{-1}\left(1-2p; \frac{1}{2}, \frac{1+k}{2}\right)} = \cos(x)$$

$$\operatorname{arcos}\left(\sqrt{F_{Beta}^{-1}\left(1-2p; \frac{1}{2}, \frac{1+k}{2}\right)}\right) = x$$

Note that arcos is arc-cosine, the inverse of the cosine function.

We must reflect the symmetric angle density for p≥0.5, so we have

$$x = \operatorname{arcos}\left(\sqrt{F_{Beta}^{-1}\left(1-2p; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p < 0.5,$$

$$= \pi - \operatorname{arcos}\left(\sqrt{F_{Beta}^{-1}\left(1-2[1-p]; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p \geq 0.5$$

IV.1 New Results: Identity Matrix & Gaussian Data

- So now we have for the angles distribution, under the Gaussian identity matrix, for the first time together, the pdf, cdf, and (analytic) quantile function:

$$f_X(x) = c_k \cdot \sin^k(x),\ x \in (0,\pi),\ k = 1,2,3\ldots\#\text{columns} - 1, \text{ and } c_k = \frac{\Gamma(k/2+1)}{\sqrt{\pi}\,\Gamma(k/2+1/2)}$$

$$F_X(x;k) \sim \frac{1}{2} - \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x < \frac{\pi}{2},$$

$$\sim \frac{1}{2} + \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x \geq \frac{\pi}{2}$$

$$F^{-1}(p;k) = \arccos\left(\sqrt{F_{Beta}^{-1}\left(1-2p; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p < 0.5;$$

$$= \pi - \arccos\left(\sqrt{F_{Beta}^{-1}\left(1-2[1-p]; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p \geq 0.5$$

Importantly, although often ignored in the sampling literature, note that properly adjusting for sample size, *n*, and degrees of freedom gives

$$k \leftarrow k + n - \#cols - 2$$

- Apparently the first (and only other) presentation of this quantile function result comes from an anonymous blog post in March, 2018, although it was obtained via a different derivation, which serves to further validate the result.

* See Xi’an, March, 2018 (https://stats.stackexchange.com/questions/331253/draw-n-dimensional-uniform-sample-from-a-unit-n-1-sphere-defined-by-n-1-dime/331850#331850 and https://xianblog.wordpress.com/2018/03/08/uniform-on-the-sphere-or-not/).
In the interest of proper attribution, a reference on the website to the book “The Bayesian Choice” hints that the Xi’an pseudonym is Christian Robert, a professor of Statistics at Université Paris Dauphine (PSL), Paris, France, since 2000 (https://stats.stackexchange.com/users/7224/xian).

IV.1 New Results: Identity Matrix & Gaussian Data

- The analytic quantile function now does, in fact, allow for the straightforward use of the inverse transform method for sampling, using a uniform random variate in place of "p" in its formula above, obviating the need for less efficient rejection sampling a la Makalic and Schmidt (2018), or any of the other much more complicated sampling schemes listed above.
- Not surprisingly, code implementing the inverse transform solution, without the need to reject samples, is about 10% faster than that of Makalic and Schmidt (2018), all else equal, as expected.
- For shorthand, we refer to this density (collectively, the pdf+cdf+analytic quantile function) as "C3" – "C"osine, plus the subscripts on the Gaussian hypergeometric function 2+1=3 for "C3."
- Of course, as stated previously, the identity matrix, while fundamental and useful for testing one specific hypothesis, arguably is the least interesting for general inference, as empirical correlation matrices in practice rarely if ever are characterized by non-diagonals that all are zero. Also, the presumption of multivariate Gaussian data is quite restrictive.

- Below we present a method to obtain the finite sample density of the correlation matrix, as well as its inverse ('quantile function'), under general conditions, requiring only the existence of the first two moments and positive definiteness.
- Note that this is conditional on 1. a specified or well estimated correlation matrix and 2. a specified or well estimated data generating mechanism.

IV.2 New Results: General Conditions

- Estimation is beyond scope: our starting point is with a presumably well estimated or specified/known correlation matrix. The estimation literature is rich and includes a very wide range of approaches, often overlapping and combined, including a variety of shrinkage approaches (Ledoit and Wolf, 2003, 2022, Lolas and Ying, 2021, Huang and Fryzlewicz, 2019); regularization (Friedman et al., 2008, Chen et al., 2019, Skachkov et al., 2021, Cui et al., 2016, Lam, 2020); hypersphere decomposition (Li, 2018); robust alternative Cholesky decompositions and ensembles thereof (Lu et al., 2020, Kang et al., 2019); thresholding (Goes et al., 2018, Tanioka et al., 2021) and other robust methods (Nanda et al., 2019; Serra et al., 2018, Yang and Cui, 2019; Auguin et al., 2018); extensions of DCC (Bauwens & Otranto, 2020); total positivity (Agrawal et al., 2020); sparse graphical modeling (Riccobello et al., 2022); CNN (Fang et al., 2021); factorized kernel approach (Zhang and Li, 2021); generalized autoregressive score (Stollenwerk, 2022); clustering (Begušić and Kostanjčar, 2019); weighted least squares (Li et al., 2022); reinforcement learning (Lu and Simaan, 2022); regular vines (Zhu and Welsch, 2018); coupled regularized, linear pooling, and spatial sign methods (Raninen, 2022); Bayesian and spherical approaches (Lan et al., 2020, Ghosh et al., 2021); random matrix theory (RMT) (Leidoit & Péché, 2011; Potters & Bouchaud, 2020), and more specifically, rotationally invariant estimators (RIE) (Bun et al., Bun, 2018, Bun et al., 2016) under non-stationarity (Bongiorno et al., 2021).

- In addition to a well-estimated correlation matrix, we presume a well-estimated, or specified/known, data generating mechanism. All responsibly and scientifically defined empirical methods require defining an appropriate range of application based on data structure, so this requirement is part and parcel of the standard scientific approach.

- Herein we continue with the geometric framework described above and established in Pinheiro & Bates (1996), Rebonato & Jaeckel (2000), Rapisarda et al. (2007), and Pourahmadi & Wang (2015).

IV.2 New Results: General Conditions

- As we saw above in the case of the Gaussian identity matrix, for the geometric framework all boils down to the distribution of the angle associated with each pairwise correlation.
- **Inference**:
  A crucially important characteristic of these angles distributions is that, unlike the distributions of the correlations themselves, they vary freely, remaining independent of each other (see Pourahmadi & Wang, 2015, Tsay & Pourahmadi, 2017, and Ghosh et al., 2020).
- This means that the multivariate density of angles is simply a function of the products of their individual densities.
- Consequently, not only do the angles define, deterministically, the correlation values themselves via

$$r_{i,j} = \cos\left(\theta_{i,1}\right)\cos\left(\theta_{j,1}\right) + \prod_{k=2}^{i-1}\cos\left(\theta_{i,k}\right)\cos\left(\theta_{j,k}\right)\prod_{l=1}^{k-1}\sin\left(\theta_{i,l}\right)\sin\left(\theta_{j,l}\right) + \cos\left(\theta_{j,i}\right)\prod_{l=1}^{i-1}\sin\left(\theta_{i,l}\right)\sin\left(\theta_{j,l}\right) \quad \text{for } 1 \le i < j \le n$$

  and thus each of the marginal distributions of the pairwise correlations (see Pourahmadi & Wang, 2015 and Ghosh et al., 2020), but also determine in a very straightforward way the probability of observing an entire correlation matrix (this probability is equivalent to a p-value, and described below).
- All we need now is the distribution of the angles under general conditions, just as we derived them above for the (Gaussian) identity matrix.
- This remains an open, and apparently non-trivial problem, at least analytically. The RMT literature shows that spectral distributions change notably and nontrivially based on the characteristics of the underlying data ensembles (e.g. heavy-tailedness (see Burda et al., 2004, Burda et al., 2006, Akemann et al., 2009; Abul-Magd et al., 2009, Bouchaud & Potters, 2015, Martin & Mahoney, 2018), and serial correlation (see Burda et al., 2004, 2011)), indicating that deriving a spectral distribution valid for all cases would be challenging, if at all possible. A similar derivation for angles distributions likely would be as challenging, and is not found in the extant literature.

IV.2 New Results: General Conditions

- However, while an “all-cases” analytic derivation for the angles distribution would be very useful, not having it is not a show stopper: we can proceed with nonparametric kernel density estimates of these distributions. The process has five steps:

1. Simulate samples (say, N=10k) based on the specified/known or well estimated data generating mechanism.
2. Calculate the corresponding N correlation matrices and their Cholesky factorizations, and transform each of these into a lower triangle matrix of angles, as described above on pp12-14.
3. Fit kernel densities to the p(p-1)/2 empirical angle distributions, each having N observations.
4. Generating samples from these densities in 3. is identical to perturbing the actual angle datapoints from 2. based on the fits in 3.
5. All samples from 4. are converted back to a re-parameterized Cholesky factorization per pp.12-14, and then multiplied by its transpose to obtain a set of N validly sampled correlation matrices. Positive definiteness is automatically enforced.

- Throughout the rest of this paper, we exhaustively compare various metrics based on 3. vs. those based on 4., as 3. is the empirical ‘truth’ against which we are testing the validity of the samples generated in 4. (technically, 4. and 5.). Once 3. is generated, further sampling from 4. is faster than turning to 3. again, although the two are equivalent.

- We dub the above sampling method via 4. **“NAbC” – Nonparametric Angles-based Correlations**.

IV.2 New Results: General Conditions

- All samples generated from Step 4. (i.e. NAbC) will yield only positive definite correlation matrices.

  Correlation matrix sampling distribution + quantile function via NAbC:

A. Using NAbC, we can now specify any (positive definite) correlation matrix of the same dimension p and use the densities from 4. (or 3.) empirically to obtain the unique cdf's associated* with each of its correlation cells: simply translate the correlation matrix to its matrix of angles, match each angle to the closest angle** in the corresponding empirical distribution of 4., return the associated empirical cdf.

B. Conversely, if we specify cdf's for each of the correlation cells, we can obtain the inverse ('quantile function') – i.e. the unique correlation matrix associated with those cdf's. Simply 'lookup' the angles in 4. (or 3.) associated with each specified cdf, translate the resulting matrix of angles to its re-parameterized Choleksky factor, and then multiply by its transpose to obtain the corresponding correlation matrix.

- The simple rule is this: any sampling or perturbation must be done after translating correlations or cdf's to angles to enforce positive definiteness – sampling/perturbation can never be performed on the correlations themselves as the resulting matrix almost certainly will not be positive definite.

- In either A. or B. above, the independence of the individual angles distributions leads to a novel 'distance metric' that also serves as a p-value: this is simply the probability of observing a correlation matrix at least as extreme at that specified, conditional on the one observed/estimated. Note that 'extreme' here implies deviation from the empirical mean correlation matrix, i.e. the expected value.***

* Note that because this 'look up' is based on an empirically generated density, near-singular extreme examples sometimes may not have been simulated, thus preventing a successful 'look up.' For these more extreme cases, the n=simulations must be increased to contain additional, more 'extreme' cases.

** Technically this is a match with the closest angle with a cdf at least as large as the specified cdf, but empirically, for N samples ≥ 10k, these are the same.

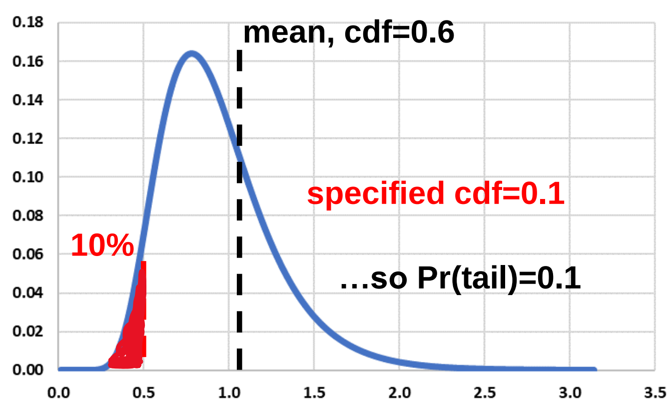
*** Note that we are only focused here on the sampling distribution, and address later in this paper the relevant issue of the non-preservation of Pearson's correlation under nonlinear transformations.

IV.2 New Results: General Conditions

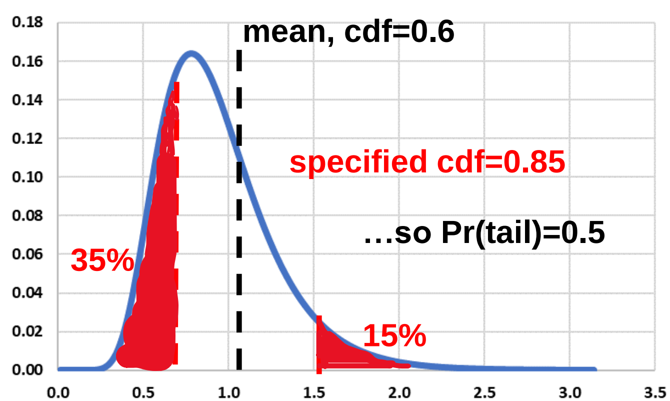
- To define the p-value, we need to calculate the empirical mean correlation matrix from the simulations in 3., and translate this to a matrix of angles. Then we obtain the empirical cdf's of these "mean-angles" with a 'look up' to the entire angles distributions from 3. Often these will be close to 0.5, but asymmetry can be notable in some cases (see APPENDICES) and must be properly accounted for.

- The p-value is calculated in two steps: determine the distance between the cdf's of the angles of the specified correlation matrix (or just the cdf's if only these are specified) and those of the "mean-angles," and sum the (usually two) tails BEYOND this distance (under notable asymmetry, this is sometimes just one tail – see Graph 1 below); then multiply all of these summed tail-density values for all of the cells. This product is the p-value for the matrix. Below are the tail probabilities for two cells with the same "mean-angles" cdf's and distributions, but different specified correlation values/cdf's:

Graph 1: p-value for a single specified (more) extreme angle cdf



Graph 2: p-value for a single specified non-extreme angle cdf



- Note that while a cdf=0.1 is hardly more 'extreme' than a cdf=0.85 in absolute terms, relative to the mean cdf=0.6, it is twice as 'extreme,' i.e. twice as far from mean cdf=0.6, and associated with only 1/5 the probability of being observed.

IV.2 New Results: General Conditions

- This p-value is very useful. It can be used for hypothesis testing, monitoring correlation matrices in statistical process control, and as a distance metric which has some advantages over widely used norms (e.g. Frobenius/Euclidean, Chebyshev, and taxi norms). The 'distance' here is a probability, not an absolute distance. Norms do not recognize that an absolute distance means different things depending on the value of the particular correlation cell. A shift of 0.01 from an original correlation value of 0.5 means something very different than the same shift from 0.98, and the p-value recognizes this, giving the 'distance' of a specified correlation matrix from an estimated one probabilistic meaning.

- Note that even in relatively small matrices (say, p=10x10) this probability becomes very small very quickly (e.g. p(p – 1)/2 = 45, and taking all cells as having 0.25 tail probabilities as an example, p-value = 0.25^45 = 8.07794E-28). But because it is based on a simple product, we can take the log of the p-value by simply summing the individual logs (which for this example, ln(p-value) = –62.38).

$$\text{"LNP"} = \ln\left(p - value\right) = \ln\left(\prod_{i=1}^{q} \Pr\left(\text{tail}\right)_i\right) = \sum_{i=1}^{q} \ln\left[\Pr\left(\text{tail}\right)_i\right] \text{ where } q = p\left(p-1\right)/2$$

- Because this is a monotonic transformation, the relative rankings of different correlation matrices compared to the estimated/observed one will be preserved. And unlike norms, its scale is not dependent on sample size, although it is dependent on the dimension of the matrix, p.

- Comparisons of the empirical distribution of LNP to those of common norms are made below.

- For completeness, a norm is a distance metric defined as: $\left\| x \right\| = \left(\sum_{i=1}^{d} \left| x_i \right|^m \right)^{1/m}$

- Taxi, Frobenius/Euclidean, and Chebyshev norms correspond to m=1, 2, and ∞, respectively (with the latter → maximum[x]).

IV.2 New Results: General Conditions

- Note that as the density of the angle distribution is bounded $\theta \in (0, \pi)$, the kernel must be appropriately reflected at the boundary via: (see Silverman, 1986)

$$\text{if } \theta < 0 \text{ then } \theta \leftarrow -\theta; \text{if } \theta > \pi \text{ then } \theta \leftarrow (2\pi - \theta)$$

- Both the Gaussian kernel and the Epanechnikov kernel have been tested extensively, along with three different bandwidth estimators, *h*, from Silverman (1986):

$$f_h(\theta) = \frac{1}{N}\sum_{i=1}^{N} K_h(\theta - \theta_i) = \frac{1}{hN}\sum_{i=1}^{N} K_h([\theta - \theta_i]/h)$$

$$\text{Gaussian: } K(\theta) = \left(1/\sqrt{2\pi}\right) \cdot e^{-\theta^2/2}$$

$$\text{Epanechnikov: } K(\theta) = (3/4) \cdot (1 - \theta^2), \ |\theta| \leq 1$$

$$h = 1.06 \cdot \hat{\sigma} \cdot N^{-1/5}$$

$$h = 0.79 \cdot \text{IQR} \cdot N^{-1/5}$$

$$h = 0.9 \cdot \min(\text{IQR}/1.34, \hat{\sigma}) \cdot N^{-1/5}$$

where $\hat{\sigma}$ = sample standard deviation and IQR = sample interquartile range

- The Epanechnikov kernel with $h = 0.9 \cdot \min(\text{IQR}/1.34, \hat{\sigma}) \cdot N^{-1/5}$ shows a slight performance edge, when comparing metrics of 3. vs. 4., across a wide range of settings, all else equal. This is likely due to the fact that in many cases, its bounded sample space will more often avoid the angle boundaries.

- Bandwidths *h* typically are the most important issue when using kernels. Here, for larger matrices (e.g. p=100x100), bandwidths need to be tightened by multiplying by a factor of 0.15 to avoid a slight drift in the density based on the large number of density estimations performed (i.e. p(p – 1)/2 = 4950). Multiplying by this factor for smaller matrices does not adversely affect the density estimation in any way, so this factor always is used. For matrices much larger than p=100, a further tightening of this factor may be required.

IV.2 New Results: General Conditions

- <u>Limitations</u>: NAbC is relatively computationally intensive, but not prohibitively so. On a commodity laptop with RAM=32GB but with no multi-threading, NAbC code processes a 100x100 matrix, with N samples = 10,000, in about 2.4 hours. In a multithreaded environment, let alone one with more memory, NAbC could be applied on similarly non-small matrices in minutes.
- Applying inverse probability transform sampling using the "C3" analytic quantile function derived herein for the specific case of the Gaussian identity matrix (no Step 3. data generated), a 100x100 matrix with N samples = 10,000 takes under 25 minutes to run on the same laptop.
- Note, however, that once the angles distributions are generated in Step 3., then using Step 4. to obtain the correlation matrices for specified cdf's ('quantile function'), or obtaining cdf matrices for specified correlation matrices, as well as the LNP=ln(p-value) values, is extremely fast. For a 100x100 matrix, generating 100 of each requires only about 2 minutes in total.
- We would argue that any disadvantage associated with NAbC not being purely 'real time' on non-small matrices is outweighed by two factors: i. its generality – it can be applied under the most general conditions possible – and ii. its unmatched flexibility – it can be applied to ANY submatrix while holding all other correlation cells constant (this is discussed in the following sections).
- Another arguable limitation is that NAbC does not include estimation, only inference and scenarios, although the extant literature on estimation is very extensive.
- Aside from NAbC, we address both actual and perceived limitations in the use of Pearson's product moment correlation matrix in a later section.
- Below, we now examine NAbC results from toy cases (p=5x5) as well as p=100x100 from some of the below-listed data generating mechansims (DGMs). We focus on comparing results based on Step 3. vs. those based on Step 4. Note that the angles graphs are meant to be toggled to more accurately see differences and similarities.

IV.2 New Results: General Conditions

- We exhaustively test to ensure that Steps 3. and 4. are, in fact, the same distributions* by comparing:
  - the angles distributions of 3. vs. 4.
  - the spectral distributions of 3. vs. 4.
  - the distributions of Euclidian/Frobenius (and other) norm(s) of 3. vs. 4.
  - the distributions of LNP = ln(p-value) of 3. vs. 4.
  - the risk (VaR-based aggregated loss / economic capital) distributions of 3. vs. 4.
  - the mean empirical correlation matrices of 3. vs. 4.
  - a scatterplot and Pearson's correlation between LNP and the Euclidian/Frobenius norm (4.)
- Under the following data generating mechanisms (DGMs):
  - MVU - Multivariate Uniform
  - MVG - Multivariate Gaussian (Gaussian marginals)
  - MVGM - Multivariate Gaussian (varying marginals)
  - MVGTS - Serially correlated approximate Multivariate Gaussian
  - MVT - Multivariate student's t
  - MVTM - Multivariate student's t copula with varying marginals
  - MVTV - Multivariate Asymmetric student's t copula with varying degrees of freedom, varying marginals and varying asymmetry
  - MVTVTS - Serially correlated approximate MVTV
  - MVTVNS - Non-stationary approximate MVTVTS
  - Archimedean Copulas (Gumbel, Frank, Clayton) with Lognormal(0,1) marginals.
- See Appendix 1 for graphical and tabular results. A summary follows below.

* Note that either 3. or 4. can be used here as "NAbC": in fact, showing that they are distributionally identical validates this entire approach. We would want to use 4. simply because sampling the empirical angles densities already generated in 3., by using 4., is much faster than re-simulating them via 3.

IV.2 New Results: General Conditions

- <u>Empirical Results from NAbC – Notes (see APPENDIX 1 for Graphs/Tables)</u>:
- NOTE: All results show a comparison of Step 3. vs. Step 4. as validation of the latter EXCEPT Case A using the C3 quantile function (because no data is being simulated – the method is analytical).
- Where noted results are for p=100x100, but otherwise, for illustrative purposes all are for p=5x5 (sample size n=p+1, and n=126 for 6 months of trading days; for p=100, n=p+1 and n=252 for a year of trading days). Kernel bandwidth h="0.9" version of Silverman (1986) x 0.15. Results include distributions of the angles, eigenvalues, Euclidian/Frobenius norm, LNP, square-root-rule elliptical aggregated 'capital,'* and the differences between all the cells in the two empirical mean correlation matrices. A scatterplot and Pearson's correlation between LNP and the Euclidian/Frobenius norm also is presented. All tables of matrices are presented in lower triangle column format. The matrices include conversions from i. specified correlation matrices to cdf matrices**; ii. specified cdf matrices to unique correlation matrices; and iii. specified %CDF-shifts from the empirical mean matrix to correlation matrices (wherein each cell's %shift is the percentage of the cumulative density above or below the cdf of the mean angle). Finally, angle number is determined based on the following fill-order (the motivation for this is discussed in a later section):

Rightmost Triangle Fill Order

11					
12	7				
13	8	4			
14	9	5	2		
15	10	6	3	1	

** Note that the inverse relationship between angles cdf's and correlation values applies here: for the %CDF-shift tables, this has been reversed so that positive/negative CDF shifts correspond with increases/decreases in correlation values.

* Under multivariate ellipticity, extreme portfolio quantiles (aka Value-at-Risk, "VaR", aka "Capital") can be calculated asymptotically with the 'square root rule,' where $\text{portfolio VaR}=\text{Capital}=\sqrt{VRV^t}$ where V = vector of p VaRs (quantiles) and R = correlation matrix of dimension p (see Tao et al., 2019, and Frachot et al., 2001). While this does not hold under other distributions, its distribution under Step 3. still should be identical to that of Step 4., so they are presented for comparison purposes. All 99VaR's are set to $50m herein: when different marginals are used, they are scaled so that 99VaR = $50m.

IV.2 New Results: General Conditions

- Empirical Results from NAbC – Example Cases (see APPENDIX 1 for Graphs/Tables):

A. C3 v MVG – Gaussian Identity Matrix: side-by-side comparison of use of fully analytic C3 quantile function (no data generation) vs. a multivariate gaussian data generating mechanism (n=6, n=126)

B. MVG with all $\boldsymbol{\rho}$ = 0.2 (n=6, n=126)

C. MVG with $\boldsymbol{\rho}$ = 0.4 for one cell (#5) v. $\boldsymbol{\rho}$ = 0.4 for one factor (row 4, cells 2,5,9) (n=6)

D. MVT (df=3) with all $\boldsymbol{\rho}$ = 0.2 (n=6, n=126)

E. MVTM – MVT (df=3) with different marginals (2 LN(0,1), 3 Gamma(1,1)), all $\boldsymbol{\rho}$ = 0.2 (n=6, n=126)

F. MVTVNS ($\boldsymbol{\rho}$ = block structure, df=3,4,5,6,7, skewness=1,0.6,0,-0.6,-1, autocorrelation= -0.25,0,0.25,0.5,0.75, nonstationarity=3σ, σ/3, σ, n/3 each) (n=6, n=126)

Case F.: *R* =

1	-0.1	-0.1	0.2	0.2
-0.1	1	-0.1	0.2	0.2
-0.1	-0.1	1	0.2	0.2
0.2	0.2	0.2	1	0.5
0.2	0.2	0.2	0.5	1

G. MVTVNS (all $\boldsymbol{\rho}$ = 0.2, **p=100x100**, df=3 to 27.5 by 0.5, skewness=1 to -0.96 by 0.04, autocorrelation=0.68 to -0.3 by 0.02, nonstationarity=3σ, σ/3, σ, n/3 each) (n=101, n=252)

H. Gumbel Copula (ϕ = 1.25), as upper tail dependence is relevant in this setting, as is a test of a multivariate distribution whose construction does not directly rely on a specified correlation matrix (n=6, n=126)

IV.2 New Results: General Conditions

- <u>Empirical Results from NAbC – Summary (see APPENDIX 1 for Graphs/Tables)</u>:
- All results are distributionally identical for Steps 3. v 4., indicating validity in NAbC's methodology.
- All spectral distributions for both 3. and 4. match those defined in the RMT literature (e.g. Marchenko-Pastur distribution (Marchenko & Pastur, 1967) under the identity matrix, and otherwise where spectral distributions are derived analytically (e.g. tests not included herein were performed using Lillo & Mantegna, 2005, and Livan et al., 2011 as examples). Also the effects of heavy-tailed distributions and serial correlation appear consistent with those of the RMT literature generally (see for the former Burda et al., 2004, Burda et al., 2006, Akemann et al., 2009; Abul-Magd et al., 2009, Bouchaud & Potters, 2015, Martin & Mahoney, 2018; and for the latter, see Burda et al., 2004, 2011).
- The relationship between LNP and the Euclidian/Frobenius norm is consistent with the latter's inability to distinguish between the RELATIVE distances from the estimated/given correlation matrix: they are very similar under the Gaussian identity matrix (especially under larger sample sizes), where relative and absolute distances are more similar, but when the original correlation matrix is asymmetric or contains more extreme values, the strength of their association diminishes somewhat, as expected.
- Note that the scale of the LNP distribution is not dependent on sample size, whereas that of the Euclidean/Frobenius and other norms do change with sample size. The scales of both are functions of the dimension of the matrix.
- On tests not presented herein, LNP remains numerically robust even under p=100x100, and readily provides rankings of 'distance' for all specified correlation and cdf matrices.

IV.2 New Results: General Conditions

- For completeness, we include the pdf of the Marchenko-Pastur (1967) distribution below:

$$f(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(\lambda_- - x)}}{\lambda x} \text{ for } \lambda_- \le x \le \lambda_+, \text{ and zero otherwise, when } 0 \le \lambda \le 1,$$

$$\text{and when } \lambda > 1 \text{ we have an additional mass point of } \left(1 - \frac{1}{\lambda}\right) \text{ at } x = 0$$

$$\text{where } \lambda_\pm = \sigma^2\left(1 \pm \sqrt{\lambda}\right),\ \lambda = \frac{p}{n},\ p = \text{\#factors/rows/columns},\ n = \text{sample size},$$

$$\text{and } \sigma = 1 \text{ for correlation matrices}$$

- This analytic result overlays the empirical spectral distributions presented in the appendices.
- Note that exceptions to convergence to this celebrated distribution do exist (see Li and Yao (2018) for an example).

IV.2 New Results: General Conditions

- Empirical Results from NAbC – Summary (see APPENDIX 1 for Graphs/Tables):
- Case A: shows distributionally identical results when comparing those generated from 3. vs. those based on 4. Also, NAbC applied to multivariate gaussian data yields distributionally identical results vs. probability inverse transform sampling using the analytic C3 quantile function derived herein, thus validating the latter empirically.
- Case B: shows identical results distributionally for 3. v 4., with the expected effects for sample size and for non-zero correlations on both the spectral and angles distributions.
- Case C: demonstrates how eigen-structure/decomposition approaches are too blunt a tool for analyzing correlation matrices. We start with the identity matrix and examine the spectral distributions of two cases: one where an entire factor/row (cells 2, 5, and 9) is assigned $\boldsymbol{\rho} = 0.4$ vs. one where only a single correlation cell (#5) gets $\boldsymbol{\rho} = 0.4$. Under small sample sizes (n=p+1) they look virtually the same. The angles distributions, however, show the story very clearly, for each of the right cells, and the untouched zero cells show exact matches with both the defined pdf (sin^k) and the C3 analytic derivation. As sample size increases, as expected, the spectral distributions can more readily be distinguished, but still do not provide the unambiguous story provided by the angles distributions. Note also how cell #1 (in row 5) is (unintentionally) affected by the changes of the factor corresponding to row 4 due to the rightmost cell-change rule described on pp. 47-49. NAbC is the first method able to explicitly control that kind of unintended correlation 'contamination.'
- Case D: shows the incremental effects of heavy tails, all else equal, compared to Case B. NAbC still matches empirical 'truth' perfectly, by every distributional criteria.
- Case E: shows the incremental effects of different marginals, all else equal, compared to Case D. NAbC still matches empirical 'truth' perfectly, by every distributional criteria.

IV.2 New Results: General Conditions

- Empirical Results from NAbC – Summary (see APPENDIX 1 for Graphs/Tables):
- Case F: shows a more 'real world' example (with a block correlation matrix) including heavy tails (varying by factor), skewness (varying by factor), autocorrelation (varying by factor), and nonstationarity. We see, again, identical results distributionally for 3. v 4. across all comparison criteria. Examination of both the spectral and angles distributions in this case shows complex structure, providing further evidence that deriving an 'all-cases' analytic density for either appears to be nontrivial.
- Case G: shows Case F. but for a non-toy matrix of p=100x100 and a constant correlation matrix with $\boldsymbol{\rho}$ = 0.2. Again, NAbC still matches empirical 'truth' perfectly, by every distributional criteria.
- Case H: shows a copula with upper tail dependence (Gumbel) that is constructed without use of, reference to, or knowledge of Pearson's correlation matrix. We see, again, identical results distributionally for 3. v 4. across all comparison criteria.

IV.2 New Results: General Conditions

- Empirical Results from NAbC – Code:
- NOTE: Complete SAS/IML (v9.4) code that generates the above results and many possible specifications for a wide range of input parameters, including input .csv files (e.g. the specified cdf matrices, correlation matrices, %CDF-shift matrices, the baseline correlation matrices, eigenvalues for defining baseline correlation matrices, etc.) will be made publicly available for download on GitHub and in the forthcoming book, “The Correlation Matrix: Robust Inference and Fully Flexible Stress Testing and Scenarios for Financial Portfolios,” Elements in Quantitative Finance series, Cambridge University Press, eds. Ricardo Rebonato, PhD.
- The SAS/IML code has extensive functionality. In addition to the optionality mentioned above, correlation matrices can be specified either as correlation matrices, or defined by eigenvalues (which are converted via Givens rotations a la Davies and Higham, 2000), matrix re-orderings for targeted submatrices can be specified as an input parameter (either as a list or in a .csv file), scales of axes on graphs and kernel bandwidths on the kernel-based graphs are input parameters, the number of angles graphs generated is a parameter, etc.

V. NAbC: Robust Inference

- NAbC is more robust than its competitors in several ways. Why is this the case?
- Almost all of NAbC's competitors, whether designed for stress testing or making inferences about an observed / estimated correlation matrix, focus on its eigen structure: either its spectral distribution (i.e. the distribution of its eigen values), or they explicitly manipulate its eigen decomposition (Archakov & Hansen (2021) is a notable exception).
- Unfortunately, in practice, the larger the matrix, the more likely it is to approach singularity, i.e. non-positive definiteness (NPD).
- So these 'eigen' approaches become only as good as the matrix is 'far' away from being NPD, which for most practical purposes of financial portfolio analysis, often is not very 'far.'
- For example, Hardin et al. (2013) (responsibly) acknowledge that, in their eigen value perturbative approach, "The amount of noise that can be added to the original matrix is determined by its smallest eigenvalue. … We provide the user with … <u>a general algorithm to apply to any correlation matrix for which the smallest eigenvalue can be reasonably estimated</u>." (emphases added).
- This constraint is true of these types of approaches generally, but it is seldom acknowledged.
- Unfortunately, this eliminates some of the most widely observed correlation matrices in finance – those close to a 'spiked' covariance matrix (see Johnstone, 2001) where one or few eigenvalues dominate and the majority of eigenvalues are close to zero, i.e. not reliably estimated.

V. NAbC: Robust Inference

- Also, eigen approaches remain at the level of the factors in the portfolio (i.e. only p factors), NOT all the pairwise associations between these factors (i.e. p(p-1)/2 associations). This is the wrong level of aggregation for flexibly analyzing the correlation matrix, cell-by-cell. Using only p eigenvalues/vectors/factors simply is too blunt a tool for inference regarding all of their p(p-1)/2 pairwise associations, at the level of the correlation cell.

- So while eigen decompositions can be indispensable for things like identifying non-random effects/factors in a portfolio, they are much less so when examining the correlation matrix per se.

- **<u>NAbC for Robust Inference</u>**:
  - <u>Structurally</u>: Level of aggregation becomes important here. There are many more angles distributions than there are spectral distributions (i.e. p(p-1)/2 vs p, a factor of (p-1)/2 more). As a matrix approaches NPD, a much smaller <u>proportion</u> of angles distributions will approach degeneracy than is true for eigenvalue distributions. Consequently, the overall construction of the correlation matrix via $R = BB^T$ will remain much more stable than one based on an eigen-decomposition of $R = V\Lambda V^{-1}$ where *V* is a matrix with column eigenvectors and Λ is a diagonal matrix of the corresponding eigenvalues.
  - <u>Empirically</u>: Even if an angle distribution approaches degeneracy, all its values will simply approach 0 or **π**. But the trigonometric functions of these values are stable, and will simply approach 0 or 1. This makes $R = BB^T$ a stable calculation empirically, even if it produces an *R* that is approaching NPD. Eigenvalue/vector estimations are more numerically involved and comparatively less stable as matrices approach NPD.
  - <u>Distributionally</u>: The distributions of angles are well behaved: they are continuously differentiable (smooth), typically unimodal (or close), and clearly bounded on $\theta \in (0, \pi)$. Spectral distributions are unbounded (in the general case) and thus characterized by larger variances and less tail accuracy. They also are more complex in the general case.
- All of this adds up to a more robust basis for inference when relying on the geometric framework of angles distributions as opposed to spectral distributions. And for examination of the correlation matrix per se, **spectral distributions simply are not at the right level of aggregation**: they are indispensable for factor analysis, but utilizing the 'eigen' information on only p factors to analyze p(p-1)/2 pairwise associations of those factors simply is too blunt a tool.

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

- **Fully Flexible Stress Testing and Scenarios**:
- Many approaches to stress testing perturb the underlying distributions to the correlation matrix (e.g. Kupiec, 1998; Ng et al., 2013; Zhang et al., 2015; Packham & Woebbeking, 2019), but this is not sufficient, just as it would not be sufficient for understanding the density of any other parameter in a portfolio model.
- It is important to reemphasize here that stressing the correlation matrix directly, as a model parameter, is not inconsistent in any way with concurrently stressing the other parameters in the model and/or the underlying marginal distributions. The Bank for International Settlements (BIS) recommends doing both concurrently.
- BIS-BCBS, 2011, pp. 28-29: "However, in order to calculate stressed VaR accurately it is also necessary to stress the correlation matrix used in all VaR methodologies. … In general, most correlations tend to increase during market crises, asymptotically approaching 1.0 during periods of complete meltdown, such as occurred in 1987, 1998 and 2008. …Certain methods that could be meaningful in this context can be identified in the earlier literature on stress testing. Employing fat-tailed distributions for the risk factors and replacing the standard correlation matrix with a stressed one are two examples." (emphasis added).
- Unfortunately, when it comes to stressing the correlation matrix directly, the literature proposes many methods that exhibit demonstrable inaccuracies and deficiencies, and unlike NAbC, none are based on probabilistic control over the individual correlation cells or the entire matrix, let alone both simultaneously.

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

- In a widely circulated paper, Galeeva et al. (2007) propose several approaches, one of which is perturbing the polar angles directly via:

$$\hat{\theta}_{i,j} = \arctan\left(\tan\left(\theta_{i,j} + \frac{\pi}{2} \right)\left(1 + \sigma z_{i,j}\right) \right) + \frac{\pi}{2} \text{ where } z_{i,j} \sim N\left(0,1\right)$$

  "with the goal of generating random angles around the base angles with some distribution which is symmetric and centered around the base-correlation $\theta^0_{i,j}$ [angle] for every i.j"

- This approach has several problems. First, the choice of a Gaussian distribution here is arbitrary, and it is not proportional to the determinant of the Jacobian as described above (see Pourahmadi & Wang, 2015). We can immediately see that the proposed distribution is not a function of the angle's column position in the matrix, which is the structure imposed by the determinant of the Jacobian. Also, in the general case the distribution is, of course, not necessarily symmetric, and is often skewed.
- Another approach proposed by Galeeva et al. (2007) is to perturb the eigenvalues by the exponentiated Gaussian distribution (i.e. Lognormal), using the historical variances as parameters in the distribution. Here the choice of the Lognormal is arbitrary, and simply does not work when matrices approach NPD: with eigenvalues ≈ 0, there is nothing to perturb.
- Other approaches proposed by Galeeva et al. (2007) (e.g. bootstrapping) simply do not preserve positive definiteness. Neither does the approach proposed in So et al. (2013).
- Ho (2015) provides a approach using empirical likelihood to modify the probability weights of sample observations to construct a stress correlation matrix. While this has advantages in its statistical interpretation (in the K-L divergence sense), non-parametric estimation, and straightforward computation, it does not provide the perfect control at the correlation cell level that NAbC provides. Neither does Loland et al., (2013).

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

- Hardin et al. (2013) utilize a normalized vector of independent gaussian random variables to perturb the observed correlation matrix. While relatively straightforward to implement, a nontrivial limitation is acknowledged by the authors: "The amount of noise that can be added to the original matrix is determined by its smallest eigenvalue. … We provide the user with … a general algorithm to apply to any correlation matrix for which the smallest eigenvalue can be reasonably estimated." (emphases added)
- Unfortunately, this eliminates some of the most widely observed correlation matrices in finance – those close to a 'spiked' covariance matrix (see Johnstone, 2001) where one or few eigenvalues dominate and the majority of eigenvalues are close to zero, i.e. not reliably estimated.
- The correlation parameterization of Packham and Woebbeking (2021) does not automatically enforce positive definiteness, and 'nearest correlation' enforcement adjustments (e.g. Higham, 2002) do systematically alter both the spectral and angles distributions. It also relies on Fisher's z-transformation, which Taraldsen (2021) has shown to sometimes be inaccurate under extreme values.
- The market stress approach of Chmielowski (2014) is a notable exception to many of the limitations listed above as it remains is explicitly invariant under a change of basis of risk factors. Parlatore and Phillippon's (2022) Kalman Filter approach in the same setting systematically and quantitatively incorporates correlation priors. But both stop short of the granular, probabilistic control needed for fully flexible stress testing/scenarios.
- So none of these methods provide the highly granular, probabilistic control over the correlation matrix directly that is required for fully flexible stress and scenarios testing. NAbC provides control at the correlation cell level, with the ability to specify scenarios for ANY submatrix of the correlation matrix while holding the rest of the values constant (e.g. for flexible use in models like Black-Litterman (1991) and its many variants), by combining two crucial findings (I. and II.) below.

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

I. Pourahmadi and Wang (2015) and others show that both the marginal distributions of individual correlations in a correlation matrix, as well as the overall distribution of the entire correlation matrix, are invariant to the ordering of the rows and columns of the matrix. Respectively,

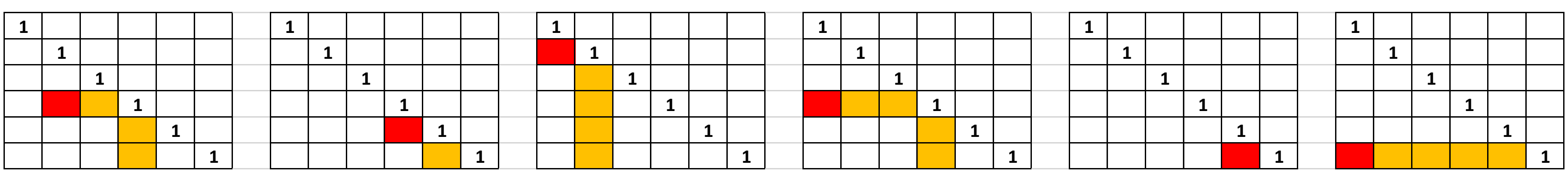
$$r_{i,j} \propto \left(1-r_{i,j}^2\right)^{[k+p/2-1]},\ i>j = Beta\left(k+\frac{p}{2},k+\frac{p}{2}\right) \text{ distribution on } (-1,1), \text{ and}$$

$$f(r)=c_p(k)\left(\prod_{j=1}^{p}\prod_{l=1}^{j-1}\left[\sin\left(\theta_{j,l}\right)\right]^2\right)^k = c_p(k)\left[\det(R)\right]^k,\ j=1,\ldots,p-1,\ i>j \quad \text{where } c_p(k)=\prod_{j=1}^{n-1}\left[\frac{\Gamma\left(\frac{2k+j}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2k+j+1}{2}\right)}\right]^j$$

II. Additionally, focusing on the lower triangle, we observe that, based on $R=BB^T$ and equivalently,

$$r_{i,j}=\cos\left(\theta_{i,1}\right)\cos\left(\theta_{j,1}\right)+\prod_{k=2}^{i-1}\cos\left(\theta_{i,k}\right)\cos\left(\theta_{j,k}\right)\prod_{l=1}^{k-1}\sin\left(\theta_{i,l}\right)\sin\left(\theta_{j,l}\right)+\cos\left(\theta_{j,i}\right)\prod_{l=1}^{i-1}\sin\left(\theta_{i,l}\right)\sin\left(\theta_{j,l}\right) \quad \text{for } 1\le i<j\le n$$

changing an arbitrary angle in *B* only will change any correlations that are to its right in the same row, and under the diagonal in the corresponding column.* Several examples are below.



* Note that some of the orange cells, and some of the rightmost triangle cells can remain unchanged if some of the corresponding correlation values are zero.

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

- Taking I. and II. together, we see that if we first reorder the correlation matrix so that only targeted cells are in the rightmost triangle of *B*, filling it in the numbered order of the matrix below, then no other cells in the matrix will be affected by changes in their values.

Rightmost Triangle Fill Order

11					
12	7				
13	8	4			
14	9	5	2		
15	10	6	3	1	

- This means that we can specify ANY combination of cells in a matrix and reorder the rows and columns accordingly so that only those correlations will be changed!

- For example, reordering the correlation matrix so that rows 1-6 are now 6-1 and columns 1-6 are now 6-1, means that the original cells 1,2 and 2,3 and 1,3 and 3,4 are now in the rightmost triangle of the lower triangular matrix, in the fill order shown above, and changes to the corresponding cells in the angles matrix B will only change these same cells, after $R = BB^T$, in the resulting correlation matrix (see below).

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

Determine Targeted Change Cells

1,2					
1,3	2,3				
		4,3			

Reorder Rows/Cols to Fill Rightmost Triangle with Targets According to Fill Order

11					
12	7				
13	8	4			
14	9	5	2		
15	10	6	3	1	

Changes in Corresponding Angles Cells ONLY change Same in Resorted Matrix

11					
12	7				
13	8	4,3			
14	9	5	2,3		
15	10	6	1,3	1,2	

- Note that the targeted cells (green) do not have to be contiguous: ANY reordering of the correlation matrix can move ANY cells to the rightmost (lower)triangle submatrix (the last column doesn't even have to be fully filled), at which point changes in these cells in the angles matrix will ONLY change these cells (orange) in the correlation matrix.

- Changes to targeted cells WILL change other cells in the targeted submatrix (due to $R = BB^T$). HOWEVER, the ordering of the submatrix matters here, and can be exploited. For example, if we want to change the 4 cells above, but subsequently want to perform 'what if' analyses on only one of those cells (e.g. cell 1,2) without changing the other three, we reorder the original correlation matrix to place that cell as the 'first' in the lower triangle of the B matrix, as shown. Then, subsequent changes to it will not affect the other (orange) cells. This 'rightmost' change rule is nested / hierarchical.

- **This gives us complete control and full flexibility in specifying scenarios; specifically, we can define which correlation cells are affected by the scenario, and which remain completely 'untouched,' all while automatically preserving positive definiteness.**

VI. NAbC: Fully Flexible Stress Testing & Scenarios for ANY Submatrix

- Yet one question remains: when implementing NAbC for targeted scenarios, what do we do with the correlation cells that we do not want to change? To keep these cells 'untouched,' we must obtain the mean empirical correlation matrix from the simulations in Step 3., translate these to a matrix of angles, and then when sampling angles via nonparametric kernel in Step 4., simply insert these mean angles as constant values in each of the appropriate cells, across all N samples.
- This way, when we convert sampled angles matrices back to correlation matrices in step 5., the correlation values in the 'untouched' cells will be their expected values,* and remain 'untouched.'
- Using the mean angles in this way not only preserves the mean correlation values,* but also still enforces positive definiteness, since we used the ANGLES from the mean empirical correlation matrix, and not the mean empirical correlation matrix itself; thus, we remain on the unit hyper-hemisphere, and remain positive definite.
- Note that other researchers have struggled with the issue of separating and isolating the effects of individual correlations without affecting others in the matrix. Ng et al. (2013) and Yu et al. (2014) identify this as changes in "peripheral" correlations due to changes in "core" correlations, and are not able to control it.
- However, the simple method described above – reorder the correlation matrix as needed, and then only modify the corresponding angles in the lower triangular matrix while filling the 'untouched' cells with the mean-angle value constant – allows for perfect control over which correlations are perturbed, and which remain completely untouched, thus solving the problem.
- The spectral distributions of these submatrix scenarios appropriately reflect their selectively limited, yet still positive definite, perturbation, as shown below in empirical results.

* Strictly from the perspective of the sampling distribution, the empirical mean correlation matrix is the right source to obtain the constant-valued angles. Note, however, that the empirical mean matrix will not match the one specified in the model under non-elliptical data. This non-preservation under non-linear transformations is discussed below.

- Empirical Results from NAbC – Example Cases, Targeted Scenario versions (APPENDIX 2):

A. **TS**: MVG – Gaussian Identity Matrix: Targeted Scenario with only first 5 (of 10) cells targeted (n=6, n=126)

F. **TS**: MVTVNS: Targeted Scenario with only first 5 (of 10) cells targeted (***ρ*** = block structure, df=3,4,5,6,7, skewness=1,0.6,0,-0.6,-1, autocorrelation=-0.25,0,0.25,0.5,0.75, nonstationarity=3σ, σ/3, σ, n/3 each) (n=6, n=126)

Case F.TS: *R* =

1	-0.1	-0.1	0.2	0.2
-0.1	1	-0.1	0.2	0.2
-0.1	-0.1	1	0.2	0.2
0.2	0.2	0.2	1	0.5
0.2	0.2	0.2	0.5	1

- Empirical Results from NAbC – Summary:

- Case A.TS: shows the effects of imposing targeted scenarios wherein only specified cells (cells 1-5) are allowed to vary. Note that the tabular results show that the values of targeted cells are essentially identical to those of Case A., as they should be, while the non-targeted cells (cells 6-10) all remain untouched (see the spikes in these angles' distributions, which are a constant value): they are the cdf's/correlation values associated with the mean empirical correlation matrix (which will be the same as the specified one under elliptical data); yet all the simulated correlation matrices that result from converting the angles matrices to Cholesky factorizations to correlation matrices all are positive definite, as required, because we remain on the hyper hemisphere by sampling angles.

- Case F.TS: shows the same results as Case A.TS under more real world data conditions. The relative effects of imposing a targeted scenario compared to Case F., the unconstrained case, are the same as those when Case A.TS is compared to Case A.

VII. Considerations in Application

- Even as its long history (see Pearson, 1895) and widespread usage and accessibility inevitably has lead to its misuse and misunderstanding in some settings, Pearson's product moment correlation, as the scaled variance-covariance matrix, remains fundamental and foundational, and it relevance and usage will remain ubiquitous. However, misuses notwithstanding, we must always consider, as with any method, its proper range of application and potential limitations under various conditions to assess whether it is fit-for-purpose for answering particular research questions in particular settings.

- <u>Linearity</u>:
The widely held view that Pearson's product moment correlation is only appropriate for measuring linear relationships is called a myth in recent research in an American Statistical Association journal. Van den Heuval & Zhan (2022) show, analytically, a number of cases where Pearson's outperforms its rank-based counterparts (e.g. Spearman's Rho and Kendall's Tau) in terms of power for detecting nonlinear monotonic associations. Conversely, they also show cases where the opposite is true for linear relationships. They conclude that defining the conditions under which one measure of association is better than another is more complex than the literature would lead us to believe, and that rejecting Pearson's correlation a priori for assessing anything but linear relationships would be ill-advised: "Pearson's correlation coefficient should not be ruled out a priori for measuring nonlinear monotonic associations.…Our examples show that existing views on linear and monotonic associations are myths." Note also that even in methodological settings characterized as highly non-linear, Pearson's often can play a central role (e.g. "Pearson Correlation Coefficient-based performance enhancement of Vanilla Neural Network for Stock Trend Prediction," Thakkar et al., 2021), as it often does in investment strategies (see Zhang et al., 2022) where it can dominate numerous other measures of association.

VII. Considerations in Application

- Non-preservation under nonlinear transformations: Unlike its rank-based counterparts, Pearson's correlation is not preserved under nonlinear transformations. For example, it is preserved in elliptical copulas (e.g. Gaussian and student's t copulas, among others), but generally not non-elliptical copulas. While it is sometimes possible to recover approximations of the original correlation values based on the pairwise empirical correlation values (see Channouf and L'Ecuyer, 2012), or generalize the two-sample problem with copulas with many parameters or using vine copulas (see Barbiero, 2019), the latter is not straightforward and remains the topic of continuing research.
- However, in many situations, it may not matter. For example, if we specify a particular Pearson's correlation for use in a non-elliptical copula (say, the asymmetric student's t-copula with varying df of Church, 2012), and its estimation fits the data very well, then for practical usage, it may not matter that the actual Pearson's correlation embedded in the model is not that specified as an input parameter. Conversely, if we were using this model to simulate data, the Pearson's correlation matrix that we would estimate from the simulated data would be different from that specified as an input, but as long as it produced the required multivariate density, it would not adversely affect the analysis.
- Still, lacking an analytical translation between true and empirical correlation matrices in the face of nonlinear transformations remains an important gap in the literature and a challenge for usage under many conditions. In the interim, to address this the SAS/IML code accompanying this monograph and to be posted on GitHub automatically generates two sets of results: one where the 'center' matrix in the sampling density is the original, specified correlation matrix, and one where this 'center' is the empirical mean correlation matrix. Under multivariate ellipticity, these two results are the same: deviations from ellipticity cause deviations in the two sets of results, and can actually be taken as a measure of departures from ellipticity. The empirical correlation matrix forms the basis of the sampling distribution under general conditions, and these are the results provided in this presentation.

VII. Considerations in Application

- Tail Risk: For assessing risk in financial portfolios, much focus is rightly placed on tail densities / dependence / co-movement, and less on the co-movement of the means (i.e. Pearson's) of the marginal distributions. It turns out, however, that under many conditions Pearson's correlation plays a direct role here as well. For elliptically distributed data, Hult and Lindskog (2002) show that the tail dependence coefficients are fully determined by only two things: the tail index, and Pearson's correlation. Similarly, Lauria et al. (2021) analytically define some tail coefficients exclusively in terms of Pearson's correlation for some semi-heavy tailed distributions (e.g. elliptical Generalized Hyperbolic). Similar to an argument one might make for using Pearson's alongside its rank-based counterparts in light of Van den Heuval & Zhan's (2022) findings, these works support the use of Pearson's as an important component of portfolio risk analysis even when the focus is on an area, i.e tail risk, that many would (wrongly) assume to be unrelated to Pearson's.

- Any quantitative method has limitations and finite ranges of appropriate application. Cavalierly using the most readily available methods, based solely on convenience and not critical vetting, as has been the case when Pearson's is misused, is antithetical to applied, scientific research. On the other hand, conveniently sweeping and overly restrictive pronouncements for or against their use even if – perhaps especially if – they are based on commonly held beliefs can make oversights in applied research more obscured and insidious, if not necessarily more misleading and damaging in the long run.

- Pearson's correlation matrix remains, at the very least, a foundational baseline. Given the current state of the literature, for most serious portfolio analyses, failing to include a rigorous examination of its potential impacts, or those of its fraternal twin, the variance-covariance matrix, would be ill advised, even if the focus is on questions not obviously directly related to them.

VIII. Summary and Conclusions

- Using a geometric framework, we have developed and presented a method defining the sampling density for the Pearson's product moment correlation matrix under the most general conditions possible: we need only the existence of the mean and the variance, and a positive definite matrix.
- The only requirements for the Nonparametric Angles-based Correlations (NAbC) method are i. a well estimated or known/specified correlation matrix and ii. a well estimated or known/specified data generating mechanism. Positive definiteness automatically is enforced on the hyper-hemisphere.
- Each unique matrix in the sampling density is associated, one-to-one, with a unique matrix of cdf's of all the angle densities associated with all the correlation cells. Given any (positive definite) matrix of the same dimensions, NAbC can provide the matrix of cdf's. NAbC also provides the inverse 'quantile function': when given a matrix of cdf's, it provides the unique, associated correlation matrix.
- Probabilities from the sampling density can be used as p-values for hypothesis tests and statistical process control; they also can be treated as a probabilistic distance metric with some advantages over norm-based distance metrics.
- <u>Fully Flexible</u>: NAbC can be applied to ANY selected cells forming ANY submatrix of the given correlation matrix, while holding all the 'untouched' cells constant, and still remain positive definite.
- This is required for fully flexible scenario specification, and not only can be used in views-based models like Black-Litterman (1991) and its variants, but also can augment and improve those views on the correlation matrix by quantifying them PROBABILISTICALLY.
- <u>Robust</u>: NAbC also is much more robust than eigen-structure approaches, remaining stable even as matrices approach singularity (where many approximations (e.g. Fisher Z-transform) breakdown).
- <u>Scalable</u>: Finally, the algorithm is scalable, providing non-prohibitive runtimes for non-small matrices (e.g. 100x100).

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IX. Next Steps / Further Research

- <u>"All-cases" Analytic Angles Distribution</u>: Analogous to deriving the angles distribution (pdf, cdf, and analytic quantile function) under the Gaussian identity matrix as is done herein, deriving the "all cases" analytic solution, under all relevant conditions, not only would add great insight to this applied research, but also speed up implementation closer to 'real time.' This open problem appears to be nontrivial, if as similarly challenging as deriving the same for spectral distributions.

- <u>Similar Research on Competing Measures</u>: Extending this analysis to Spearman's Rho and Kendall's Tau potentially would be extremely useful. The latter already has been shown to have more extensive connections to Pearson's correlation than previously thought (see Lindskog et al., 2003), and some recent results show similar (see Bandeira et al., 2017, and Li, Wang, and Li, 2021), but not identical (see Li, Wang, and Wang, 2021), behavior in their spectral distributions.

- <u>Comparisons Against / Conjoint Usage with, Competing Methods</u>: Comparing NAbC, under wide-ranging test conditions, to Hansen & Archakov (2021) and the Bayesian approaches of Lan et al. (2020) and Ghosh et al. (2020) likely would be a useful and insightful exercise. The four approaches differ in notable ways, yet have some strong similarities: the two Bayesian approaches both adopt the hyperspherical geometric framework that forms the foundation for NAbC. Perhaps using them in conjunction would be fruitful, as the limitations of each cited above may be the strengths of the other. This certainly appears to be the case for the evolutionary approach of Papenbrock et al. (2021).

- <u>Statistical Process Monitoring</u>: While NAbC's application to hypothesis testing is self-evident, it would be useful to see if the many statistical process control methods designed for monitoring the correlation and covariance matrices could make use of the NAbC sampling density, or if the latter could be used to validate results of the methods proposed in papers like Adegoke et al. (2022), Ajadi et al. (2021), Bours & Steland (2020), Wang et al. (2019), Choi and Shin (2021), and others like those reviewed in Ebadi et al. (2021).

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XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

- Empirical Results from NAbC – Notes:
- NOTE: All results show a comparison of Step 3. vs. Step 4. as validation of the latter EXCEPT Case A using the C3 quantile function (because no data is being simulated – the method is analytical).
- Where noted results are for p=100x100, but otherwise, for illustrative purposes all are for p=5x5 (sample size n=p+1, and n=126 for 6 months of trading days; for p=100, n=p+1 and n=252 for a year of trading days). Kernel bandwidth h="0.9" version of Silverman (1986) x 0.15. Results include distributions of the angles, eigenvalues, Euclidian/Frobenius norm, LNP, square-root-rule elliptical aggregated 'capital,'* and the differences between all the cells in the two empirical mean correlation matrices. A scatterplot and Pearson's correlation between LNP and the Euclidian/Frobenius norm also is presented. All tables of matrices are presented in lower triangle column format. The matrices include conversions from i. specified correlation matrices to cdf matrices**; ii. specified cdf matrices to unique correlation matrices; and iii. specified %CDF-shifts from the empirical mean matrix to correlation matrices (wherein each cell's %shift is the percentage of the cumulative density above or below the cdf of the mean angle). Finally, angle number is determined based on the following fill-order (the motivation for this is discussed in a later section):

Rightmost Triangle Fill Order

11					
12	7				
13	8	4			
14	9	5	2		
15	10	6	3	1	

** Note that the inverse relationship between angles cdf's and correlation values applies here: for the %CDF-shift tables, this has been reversed so that positive/negative CDF shifts correspond with increases/decreases in correlation values.

* Under multivariate ellipticity, extreme portfolio quantiles (aka Value-at-Risk, "VaR", aka "Capital") can be calculated asymptotically with the 'square root rule,' where $\text{portfolio VaR=Capital} = \sqrt{VRV^t}$ where V = vector of p VaRs (quantiles) and R = correlation matrix of dimension p (see Tao et al., 2019, and Frachot et al., 2001). While this does not hold under other distributions, its distribution under Step 3. still should be identical to that of Step 4., so they are presented for comparison purposes. All 99VaR's are set to $50m herein: when different marginals are used, they are scaled so that 99VaR = $50m.

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

- Empirical Results from NAbC – Example Cases:

A. C3 v MVG – Gaussian Identity Matrix: side-by-side comparison of use of fully analytic C3 quantile function (no data generation) vs. a multivariate gaussian data generating mechanism (n=6, n=126)

B. MVG with all $\boldsymbol{\rho}$ = 0.2 (n=6, n=126)

C. MVG with $\boldsymbol{\rho}$ = 0.4 for one cell (#5) v. $\boldsymbol{\rho}$ = 0.4 for one factor (row 4, cells 2,5,9) (n=6)

D. MVT (df=3) with all $\boldsymbol{\rho}$ = 0.2 (n=6, n=126)

E. MVTM – MVT (df=3) with different marginals (2 LN(0,1), 3 Gamma(1,1)), all $\boldsymbol{\rho}$ = 0.2 (n=6, n=126)

F. MVTVNS ($\boldsymbol{\rho}$ = block structure, df=3,4,5,6,7, skewness=1,0.6,0,-0.6,-1, autocorrelation= -0.25,0,0.25,0.5,0.75, nonstationarity=3σ, σ/3, σ, n/3 each) (n=6, n=126)

Case F.: *R* =

1	-0.1	-0.1	0.2	0.2
-0.1	1	-0.1	0.2	0.2
-0.1	-0.1	1	0.2	0.2
0.2	0.2	0.2	1	0.5
0.2	0.2	0.2	0.5	1

G. MVTVNS (all $\boldsymbol{\rho}$ = 0.2, **p=100x100**, df=3 to 27.5 by 0.5, skewness=1 to -0.96 by 0.04, autocorrelation=0.68 to -0.3 by 0.02, nonstationarity=3σ, σ/3, σ, n/3 each) (n=101, n=252)

H. Gumbel Copula (ϕ = 1.25), as upper tail dependence is relevant in this setting, as is a test of a multivariate distribution whose construction does not directly rely on a specified correlation matrix (n=6, n=126)

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

- Empirical Results from NAbC – Summary:
- All results are distributionally identical for Steps 3. v 4., indicating validity in NAbC's methodology.
- All spectral distributions for both 3. and 4. match those defined in the RMT literature (e.g. Marchenko-Pastur distribution (Marchenko & Pastur, 1967) under the identity matrix, and otherwise where spectral distributions are derived analytically (e.g. tests not included herein were performed using Lillo & Mantegna, 2005, and Livan et al., 2011 as examples). Also the effects of heavy-tailed distributions and serial correlation appear consistent with those of the RMT literature generally (see for the former Burda et al., 2004, Burda et al., 2006, Akemann et al., 2009; Abul-Magd et al., 2009, Bouchaud & Potters, 2015, Martin & Mahoney, 2018; and for the latter, see Burda et al., 2004, 2011).
- The relationship between LNP and the Euclidian/Frobenius norm is consistent with the latter's inability to distinguish between the RELATIVE distances from the estimated/given correlation matrix: they are very similar under the Gaussian identity matrix (especially under larger sample sizes), where relative and absolute distances are more similar, but when the original correlation matrix is asymmetric or contains more extreme values, the strength of their association diminishes somewhat, as expected.
- Note that the scale of the LNP distribution is not dependent on sample size, whereas that of the Euclidean/Frobenius and other norms do change with sample size. The scales of both are functions of the dimension of the matrix.
- On tests not presented herein, LNP remains numerically robust even under p=100x100, and readily provides rankings of 'distance' for all specified correlation and cdf matrices.

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

- For completeness, we include the pdf of the Marchenko-Pastur (1967) distribution below:

$$f(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(\lambda_- - x)}}{\lambda x} \text{ for } \lambda_- \le x \le \lambda_+, \text{ and zero otherwise, when } 0 \le \lambda \le 1,$$

$$\text{and when } \lambda > 1 \text{ we have an additional mass point of } \left(1 - \frac{1}{\lambda}\right) \text{ at } x = 0$$

$$\text{where } \lambda_\pm = \sigma^2\left(1 \pm \sqrt{\lambda}\right),\ \lambda = \frac{p}{n},\ p = \text{\#factors/rows/columns},\ n = \text{sample size,}$$

$$\text{and } \sigma = 1 \text{ for correlation matrices}$$

- This analytic result overlays the empirical spectral distributions presented in the appendices.
- Note that exceptions to convergence to this celebrated distribution do exist (see Li and Yao (2018) for an example).

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

- Empirical Results from NAbC – Summary:
- Case A: shows distributionally identical results when comparing those generated from 3. vs. those based on 4. Also, NAbC applied to multivariate gaussian data yields distributionally identical results vs. probability inverse transform sampling using the analytic C3 quantile function derived herein, thus validating the latter empirically.
- Case B: shows identical results distributionally for 3. v 4., with the expected effects for sample size and for non-zero correlations on both the spectral and angles distributions.
- Case C: demonstrates how eigen-structure/decomposition approaches are too blunt a tool for analyzing correlation matrices. We start with the identity matrix and examine the spectral distributions of two cases: one where an entire factor/row (cells 2, 5, and 9) is assigned $\boldsymbol{\rho} = 0.4$ vs. one where only a single correlation cell (#5) gets $\boldsymbol{\rho} = 0.4$. Under small sample sizes (n=p+1) they look virtually the same. The angles distributions, however, show the story very clearly, for each of the right cells, and the untouched zero cells show exact matches with both the defined pdf (sin^k) and the C3 analytic derivation. As sample size increases, as expected, the spectral distributions can more readily be distinguished, but still do not provide the unambiguous story provided by the angles distributions. Note also how cell #1 (in row 5) is (unintentionally) affected by the changes of the factor corresponding to row 4 due to the rightmost cell-change rule described on pp. 47-49. NAbC is the first method able to explicitly control that kind of unintended correlation 'contamination.'
- Case D: shows the incremental effects of heavy tails, all else equal, compared to Case B. NAbC still matches empirical 'truth' perfectly, by every distributional criteria.
- Case E: shows the incremental effects of different marginals, all else equal, compared to Case D. NAbC still matches empirical 'truth' perfectly, by every distributional criteria.

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

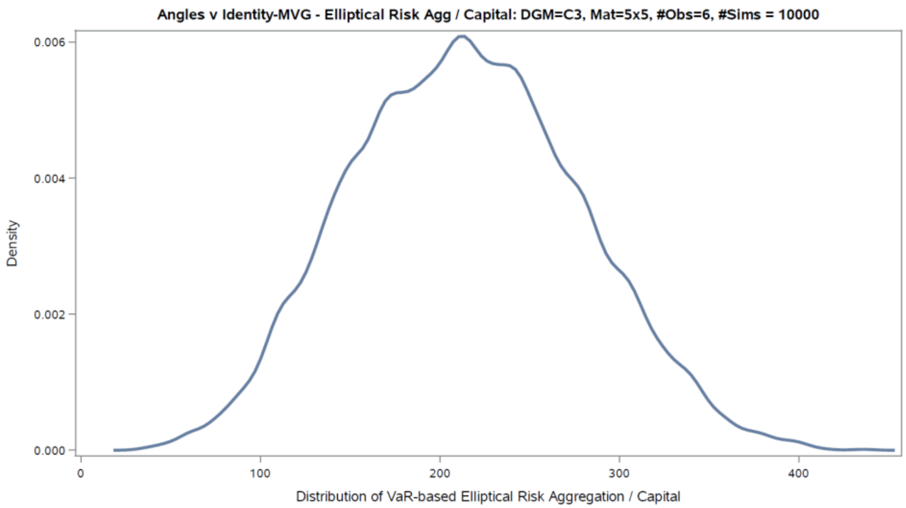
- Empirical Results from NAbC – Summary:
- Case F: shows a more 'real world' example (with a block correlation matrix) including heavy tails (varying by factor), skewness (varying by factor), autocorrelation (varying by factor), and nonstationarity. We see, again, identical results distributionally for 3. v 4. across all comparison criteria. Examination of both the spectral and angles distributions in this case shows complex structure, providing further evidence that deriving an 'all-cases' analytic density for either appears to be nontrivial.
- Case G: shows Case F. but for a non-toy matrix of p=100x100 and a constant correlation matrix with $\boldsymbol{\rho}$ = 0.2. Again, NAbC still matches empirical 'truth' perfectly, by every distributional criteria.
- Case H: shows a copula with upper tail dependence (Gumbel) that is constructed without use of, reference to, or knowledge of Pearson's correlation matrix. We see, again, identical results distributionally for 3. v 4. across all comparison criteria.

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix

- Empirical Results from NAbC – Code:
- NOTE: Complete SAS/IML (v9.4) code that generates the above results and many possible specifications for a wide range of input parameters, including input .csv files (e.g. the specified cdf matrices, correlation matrices, %CDF-shift matrices, the baseline correlation matrices, eigenvalues for defining baseline correlation matrices, etc.) will be made publicly available for download on GitHub and in the forthcoming book, "The Correlation Matrix: Robust Inference and Fully Flexible Stress Testing and Scenarios for Financial Portfolios," Elements in Quantitative Finance series, Cambridge University Press, eds. Ricardo Rebonato, PhD.
- The SAS/IML code has extensive functionality. In addition to the optionality mentioned above, correlation matrices can be specified either as correlation matrices, or defined by eigenvalues (which are converted via Givens rotations a la Davies and Higham, 2000), matrix re-orderings for targeted submatrices can be specified as an input parameter (either as a list or in a .csv file), scales of axes on graphs and kernel bandwidths on the kernel-based graphs are input parameters, the number of angles graphs generated is a parameter, etc.

Elliptical Capital

C3 (Analytic) **Sample Size n = 6** **MVG Data**



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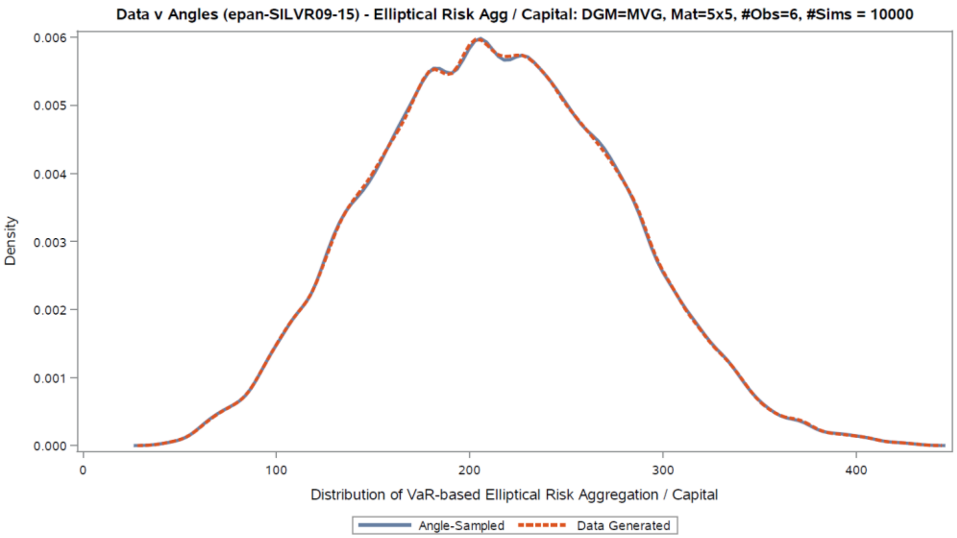

PRNG Seed = 12345
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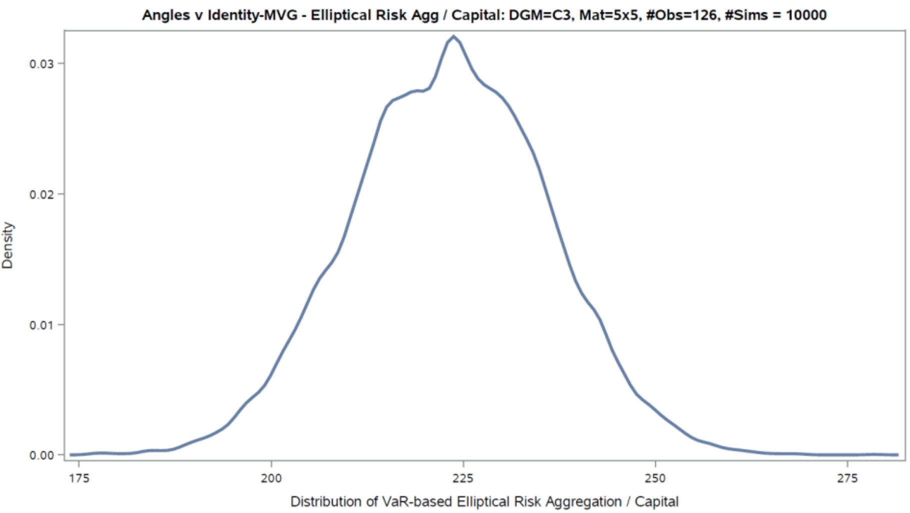
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

C3 (Analytic) | **Elliptical Capital Sample Size n = 126** | **MVG Data**



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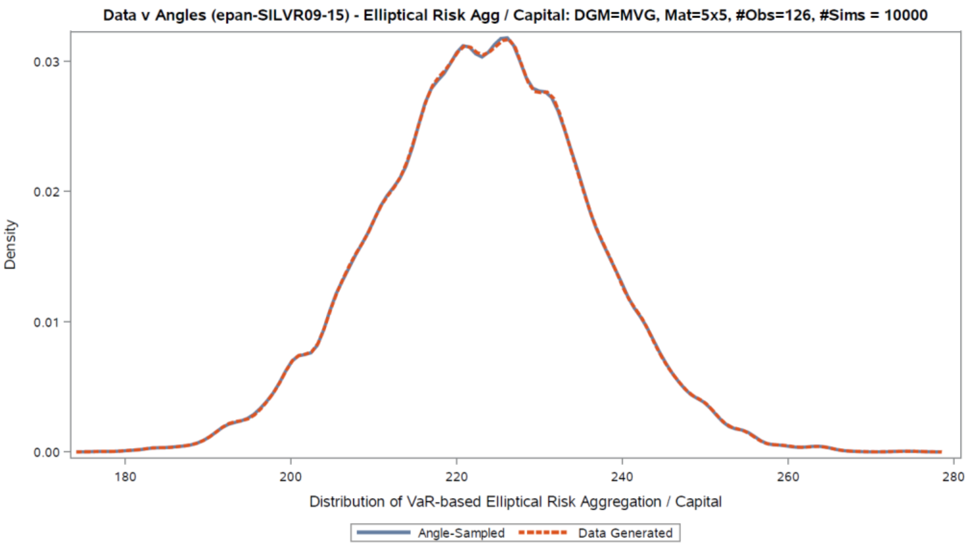

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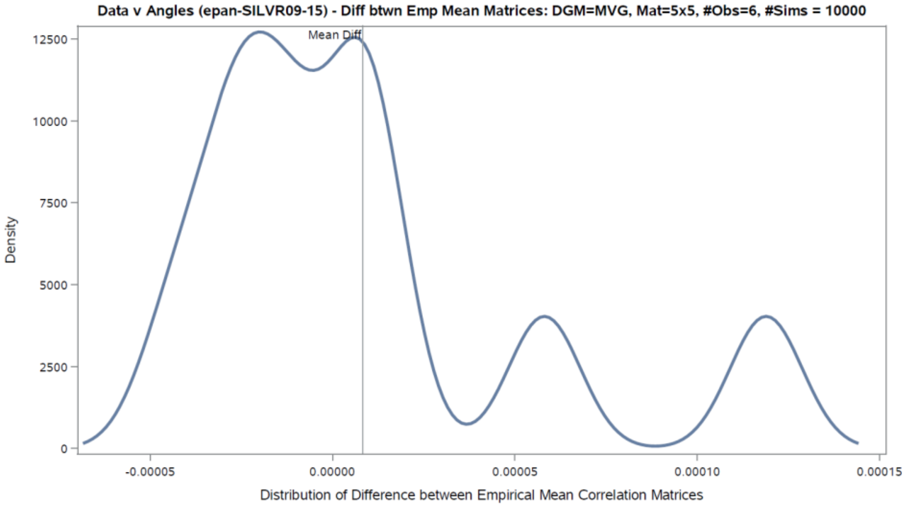
Difference Between Mean Empirical Matrices

C3 (Analytic) **Sample Size n = 6** **MVG Data**

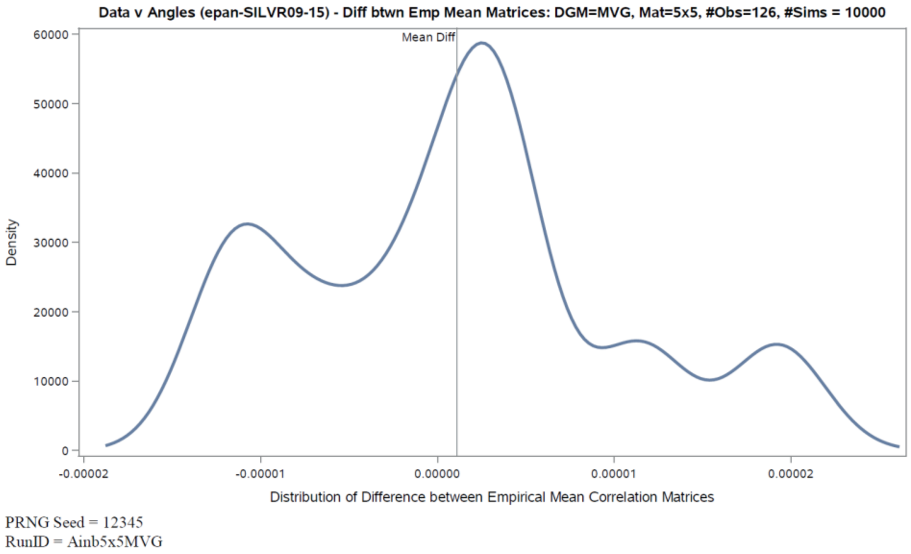


PRNG Seed = 12345
RunID = Ains5x5MVG

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Difference Between Mean Empirical Matrices

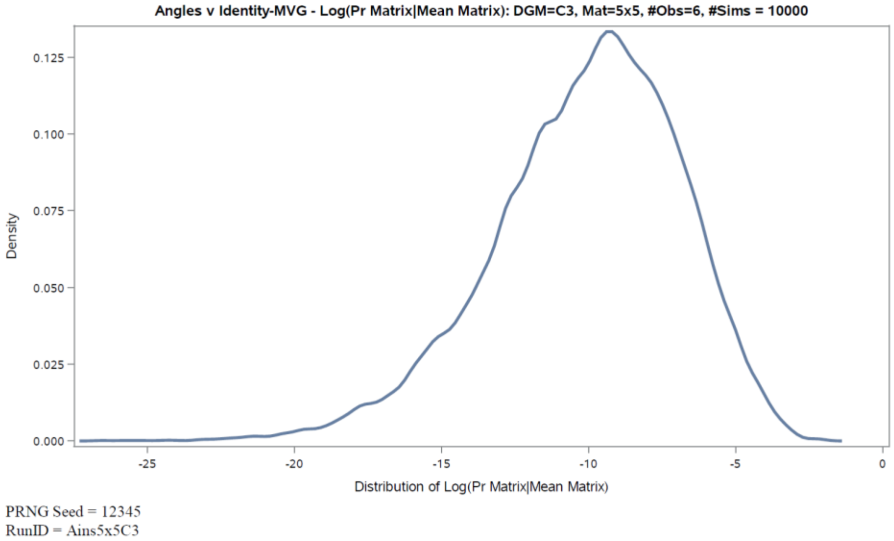
C3 (Analytic) Sample Size n = 126 MVG Data

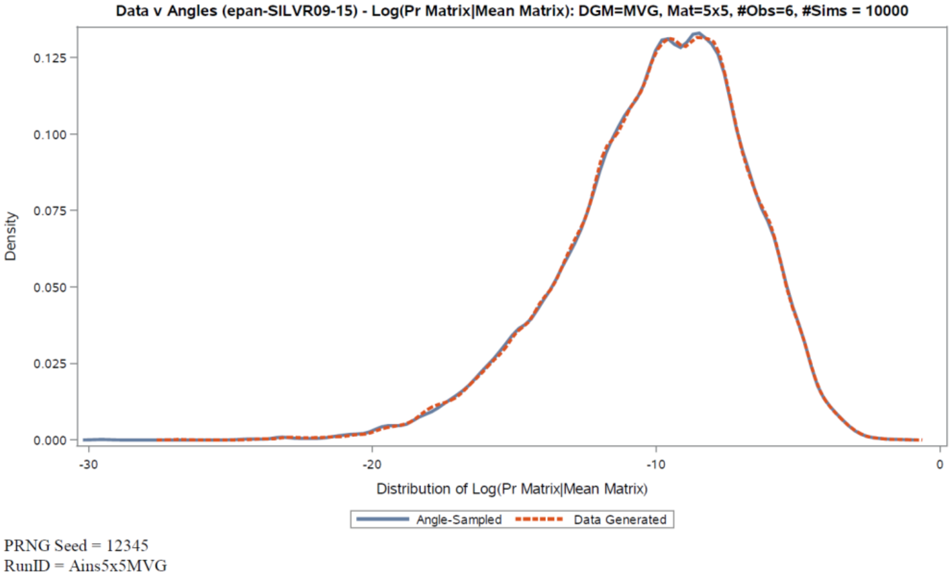


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

LNP

C3 (Analytic) Sample Size n = 6 MVG Data

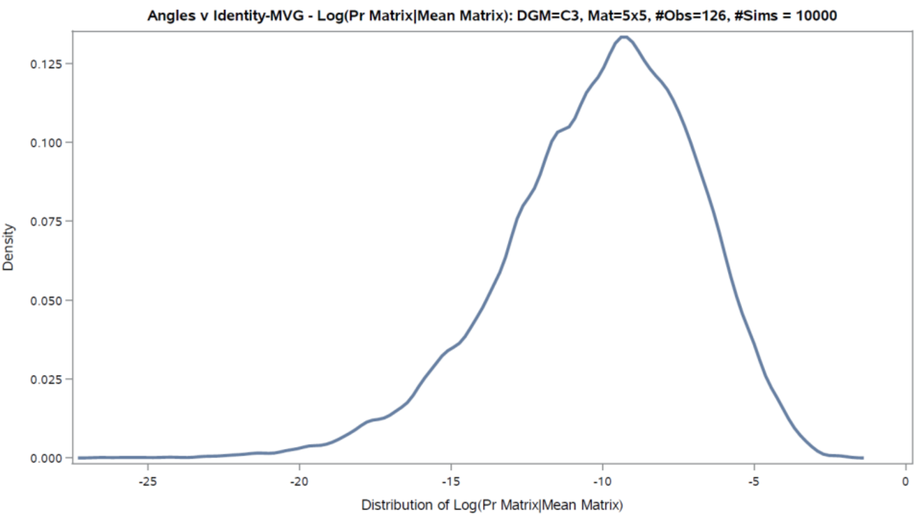




XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

LNP

C3 (Analytic) **Sample Size n = 126** **MVG Data**



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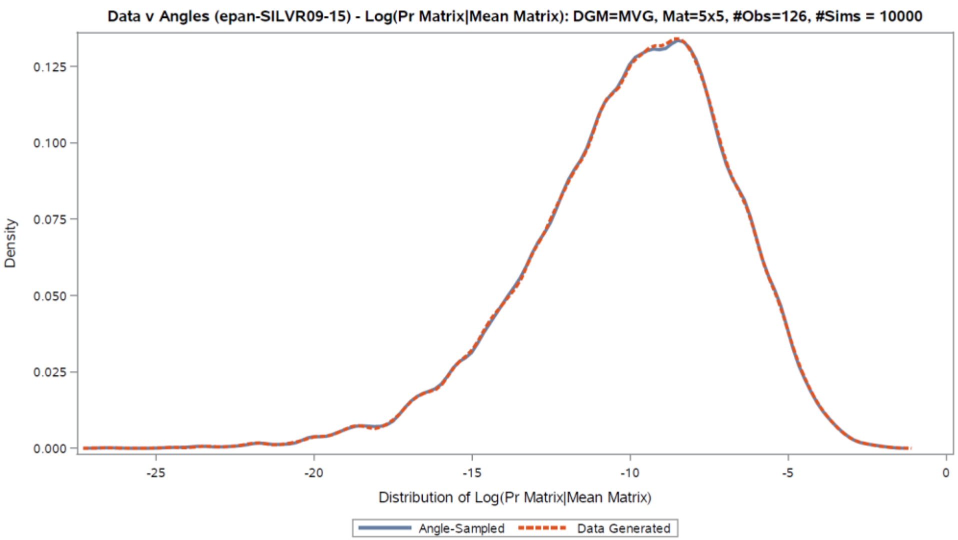

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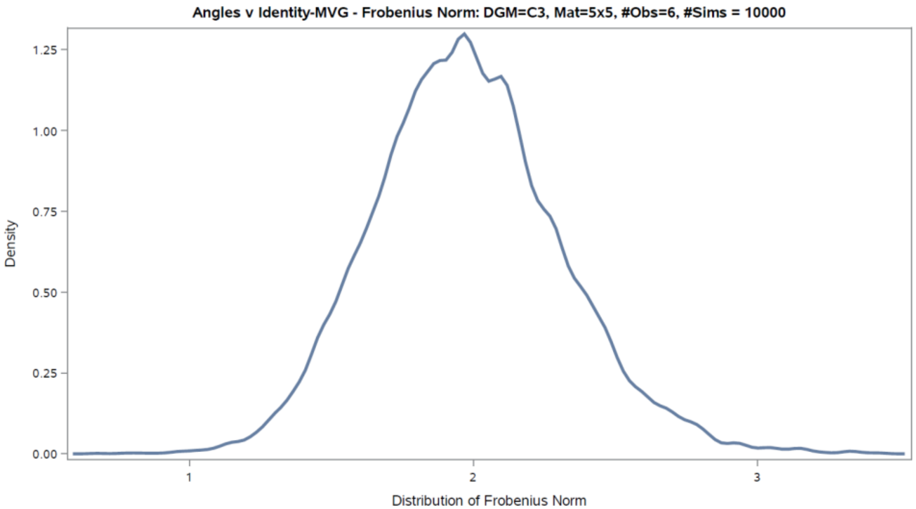
Euclidian/Frobenius Norm

C3 (Analytic) **Sample Size n = 6** **MVG Data**



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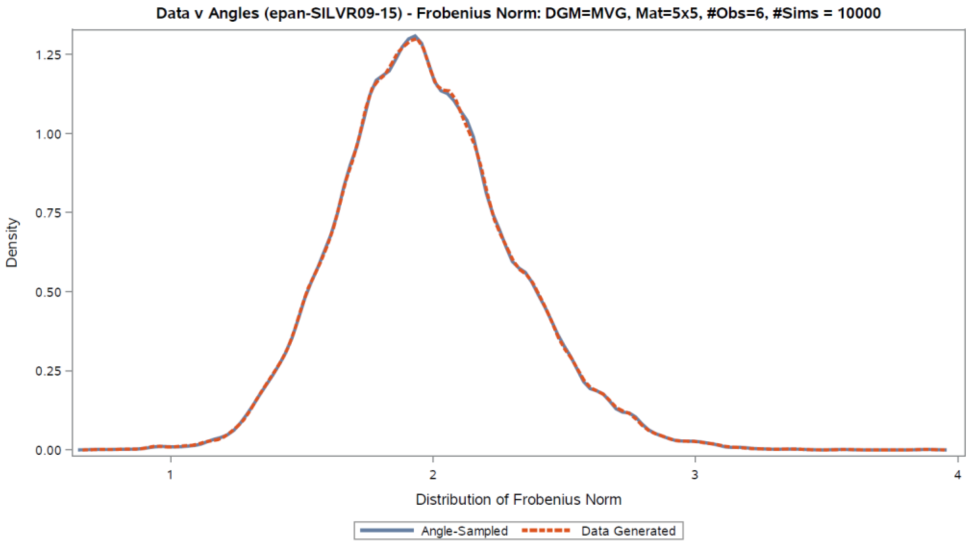

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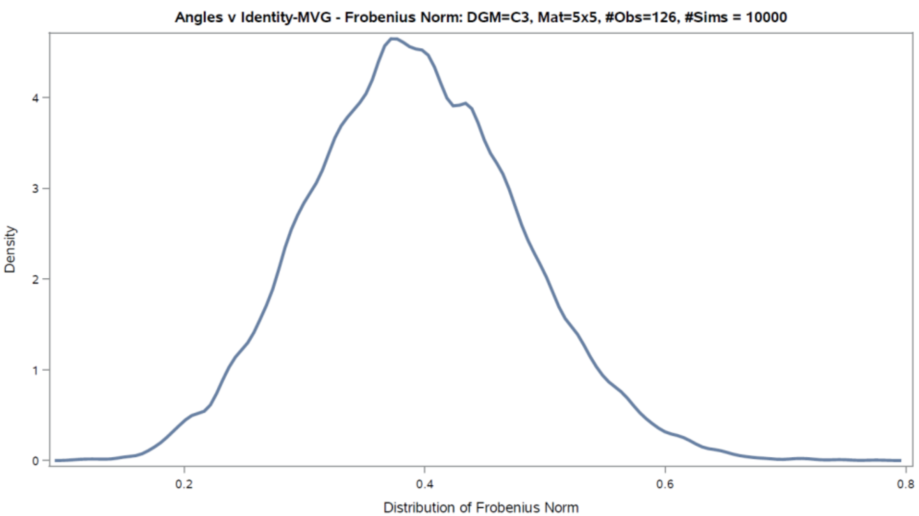
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Euclidian/Frobenius Norm

C3 (Analytic) **Sample Size n = 126** **MVG Data**



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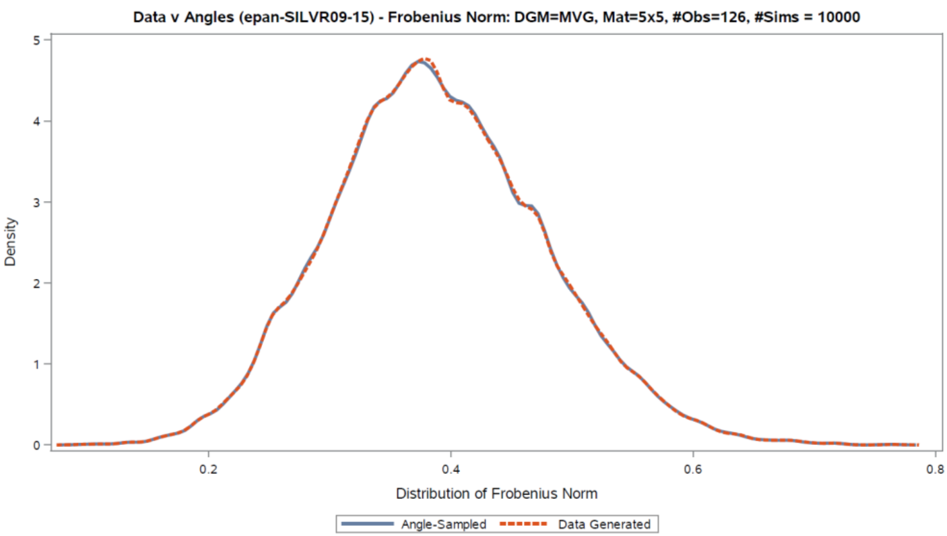

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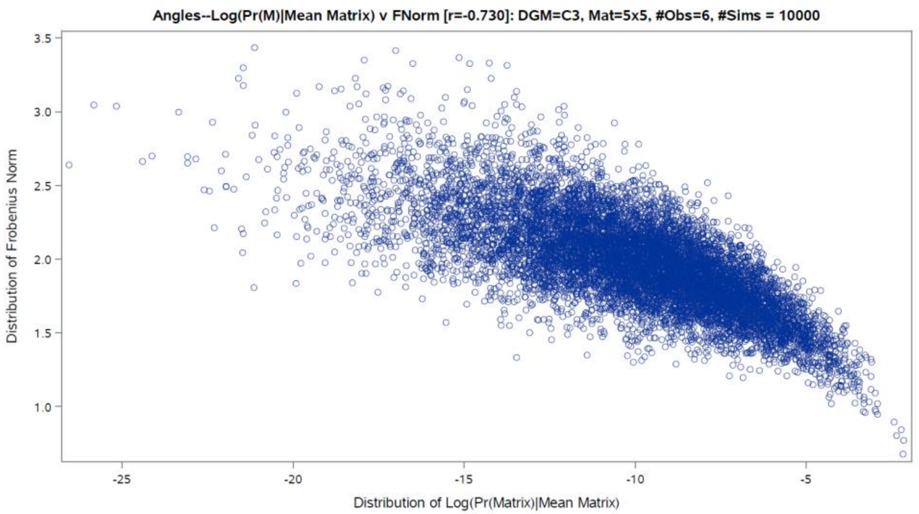
LNP v Euclidian/Frobenius Norm

C3 (Analytic) **Sample Size n = 6** **MVG Data**



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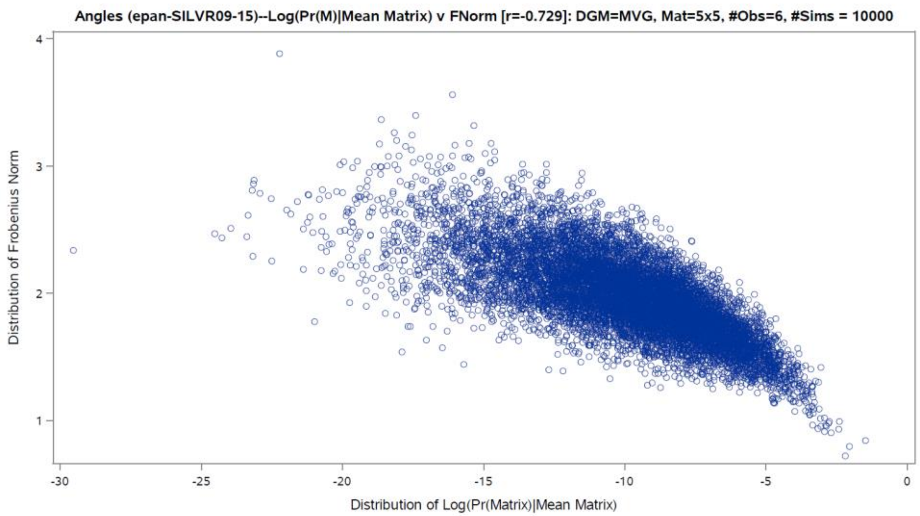

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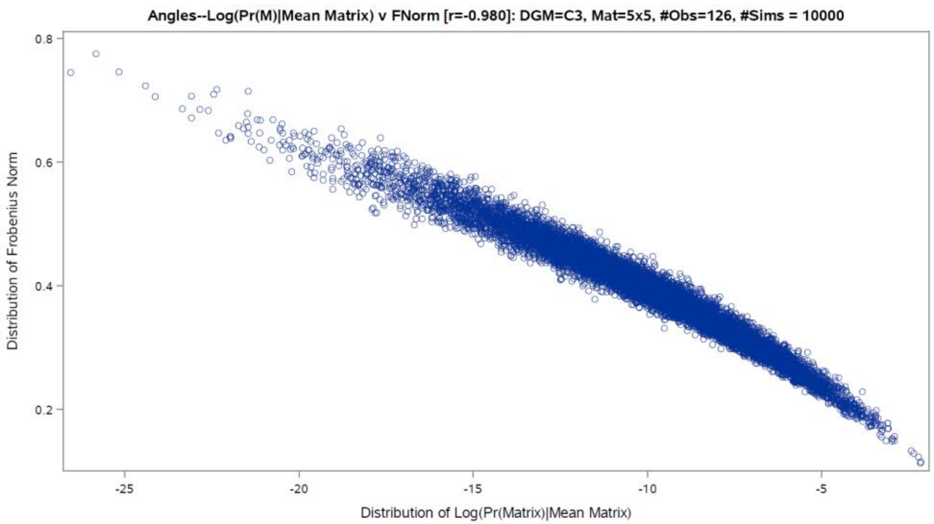
LNP v Euclidian/Frobenius Norm

C3 (Analytic) **Sample Size n = 126** **MVG Data**



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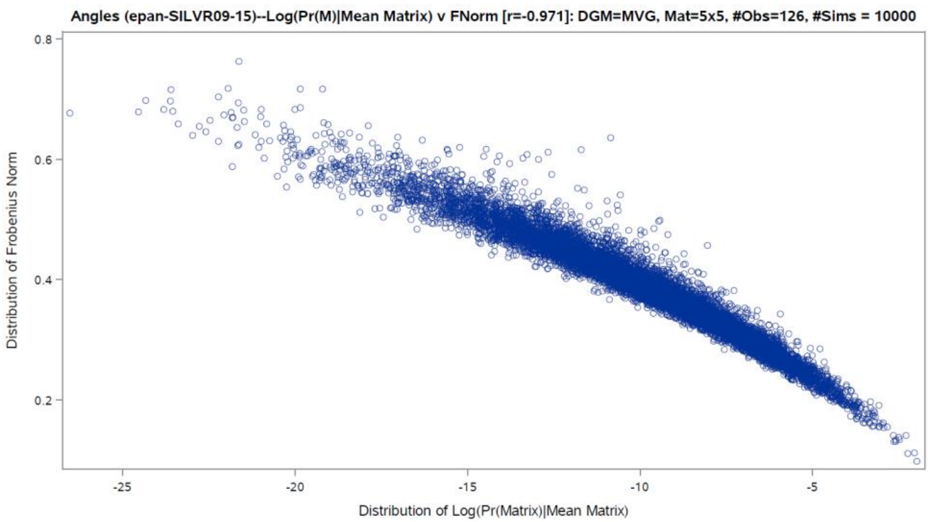

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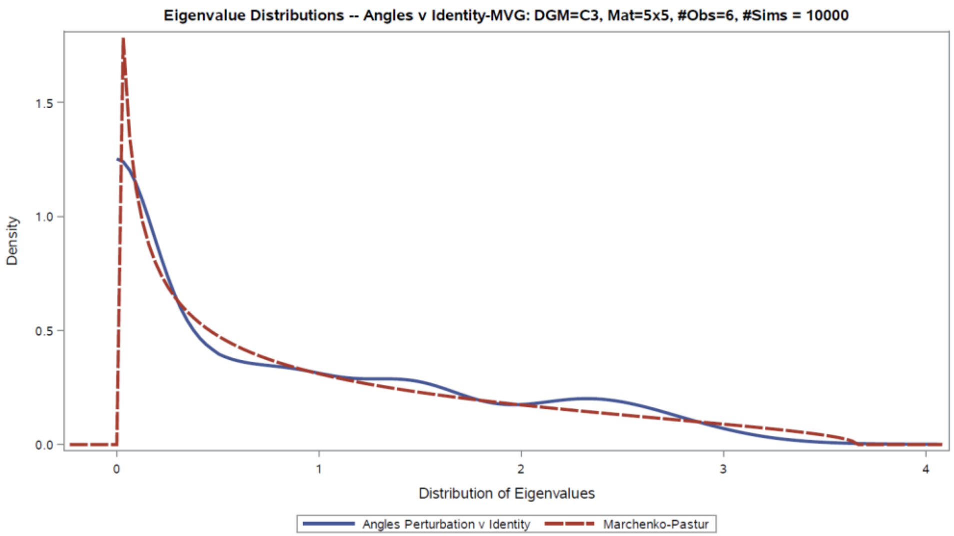
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Spectral Distributions

C3 (Analytic) **Sample Size n = 6** **MVG Data**



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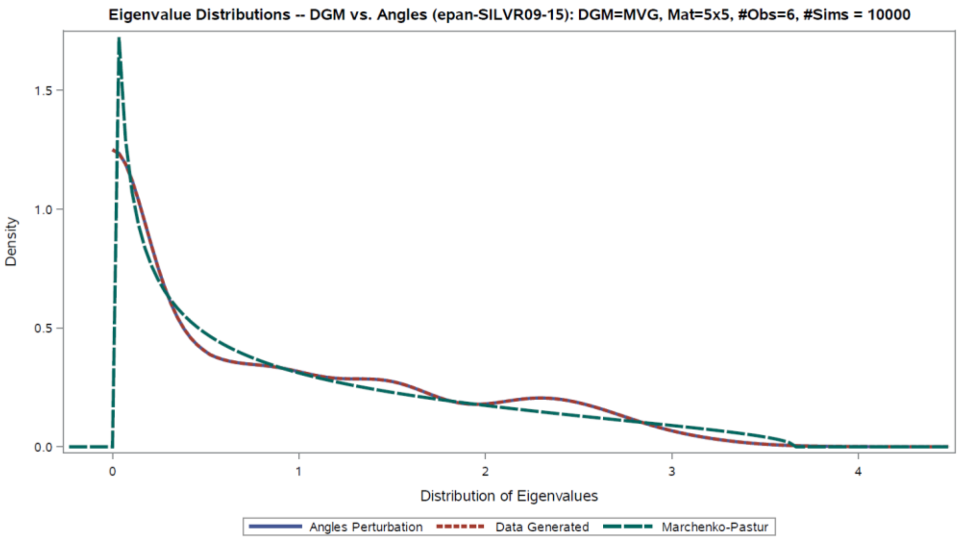

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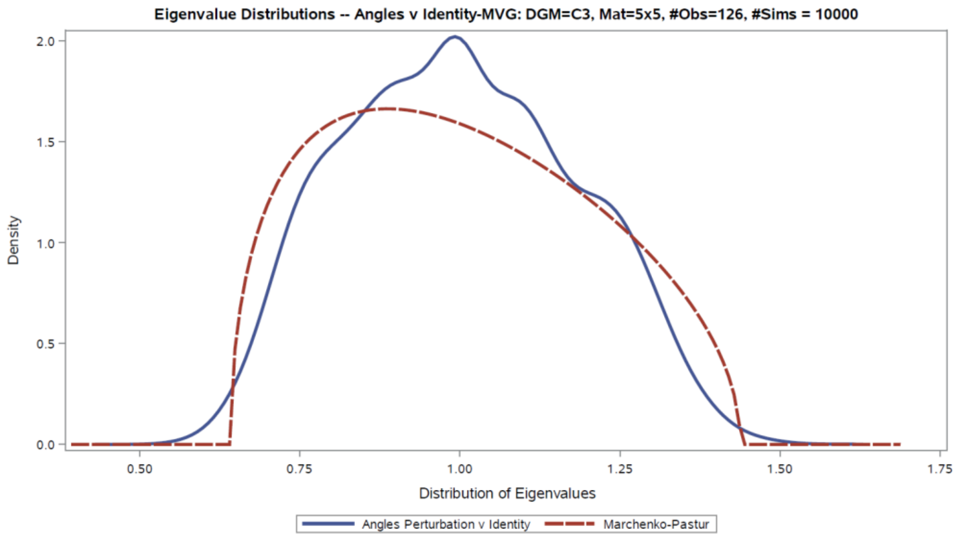
Spectral Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



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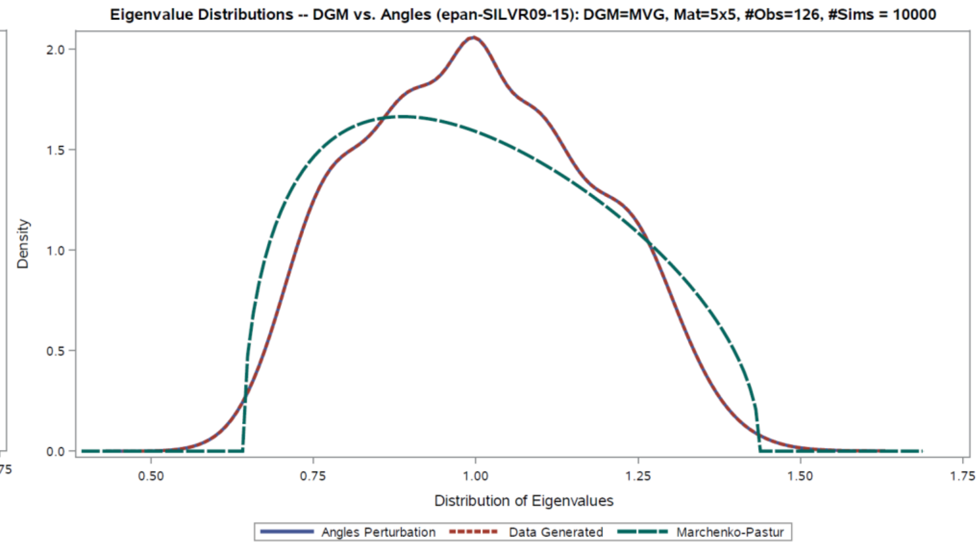

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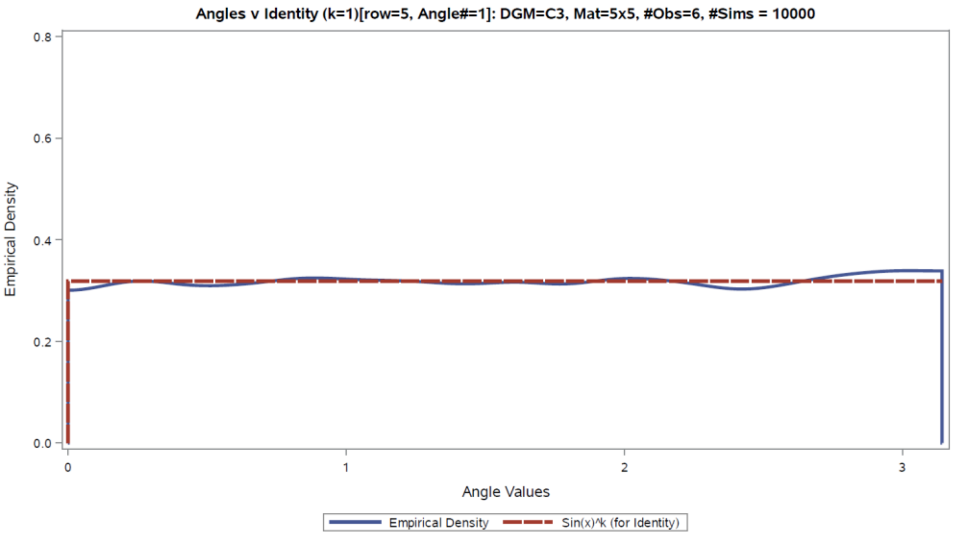
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) Sample Size n = 6 MVG Data



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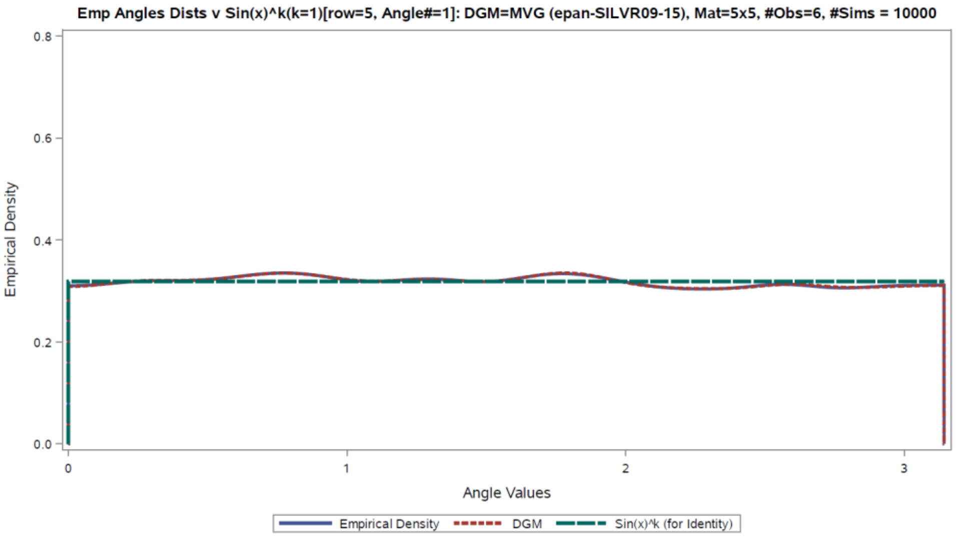

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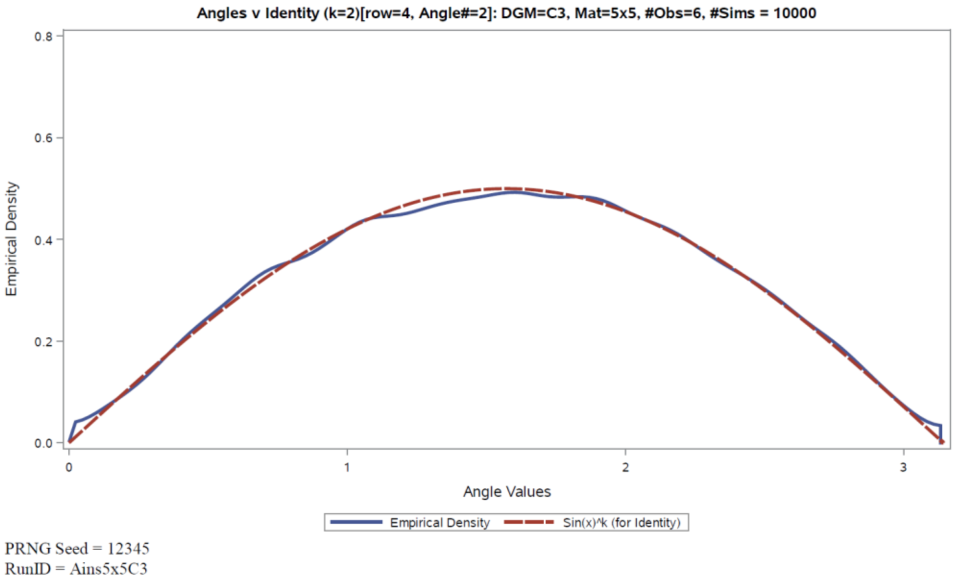
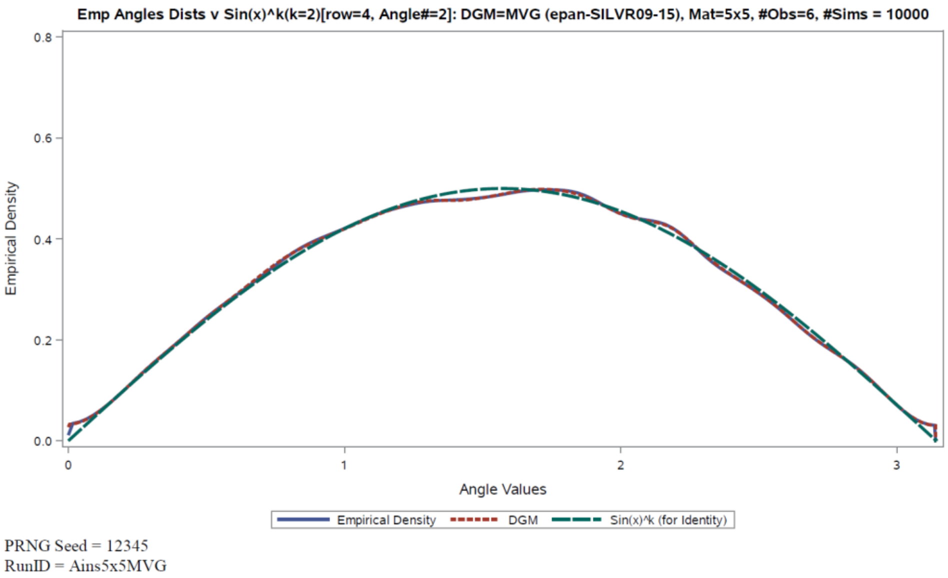
Angle Distributions

C3 (Analytic) **Sample Size n = 6** **MVG Data**

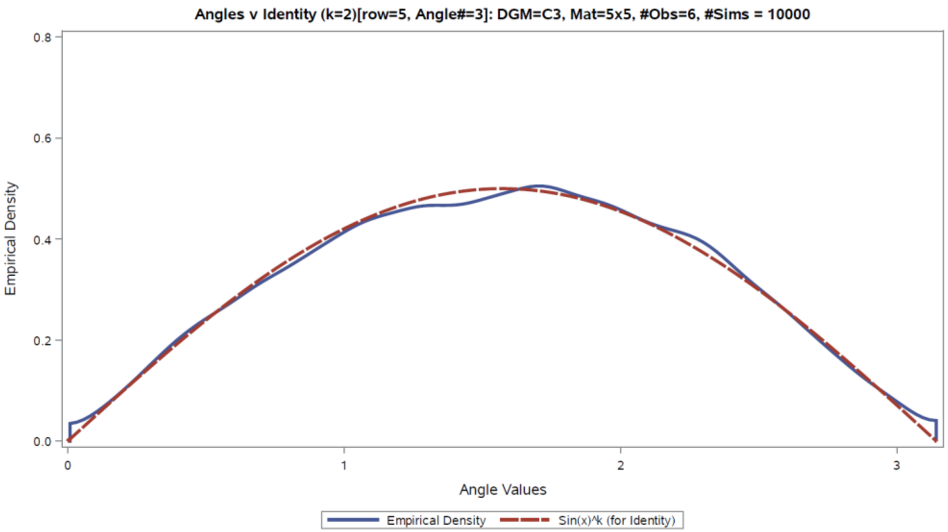




XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) **Sample Size n = 6** **MVG Data**



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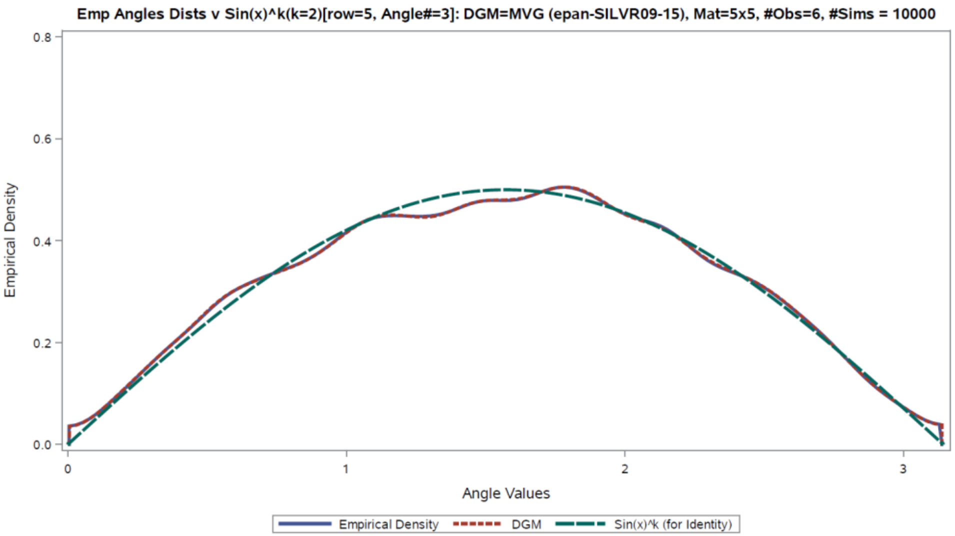

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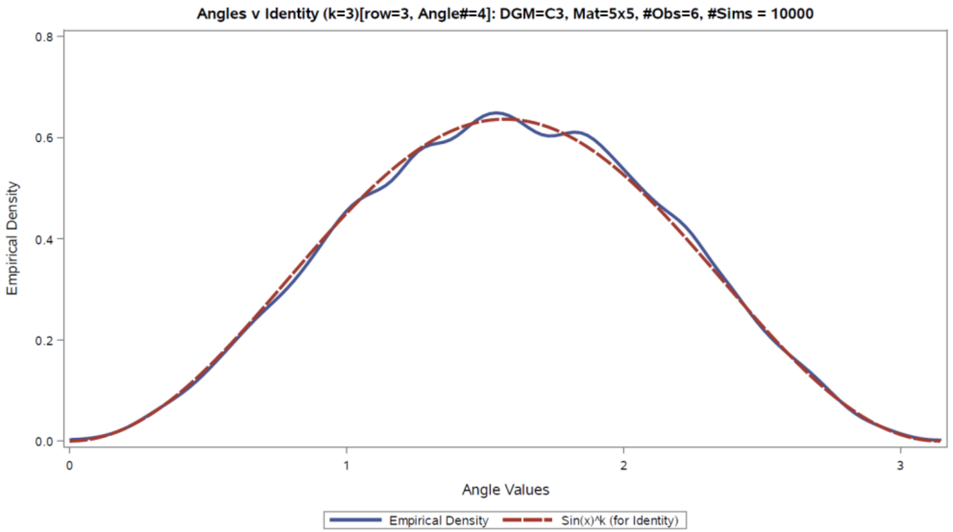
Angle Distributions

C3 (Analytic) — **Sample Size n = 6** — **MVG Data**



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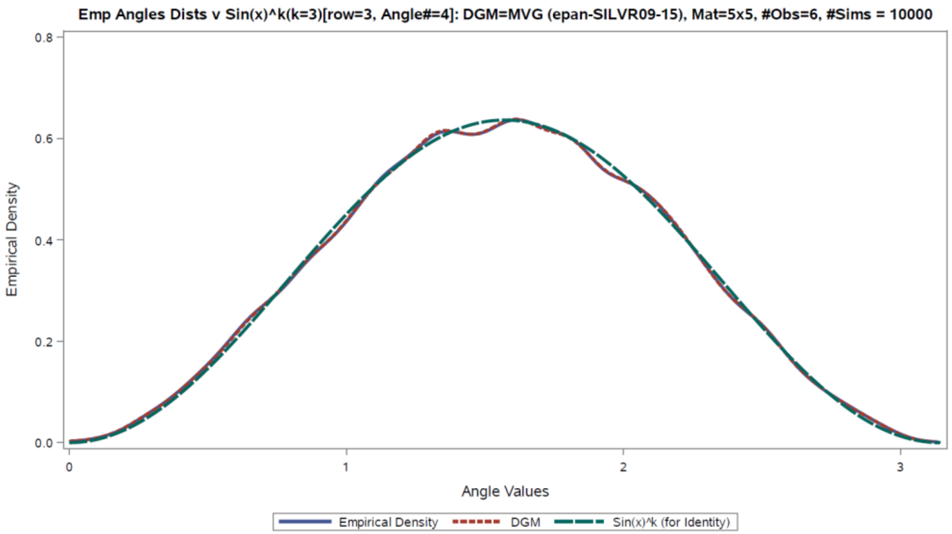

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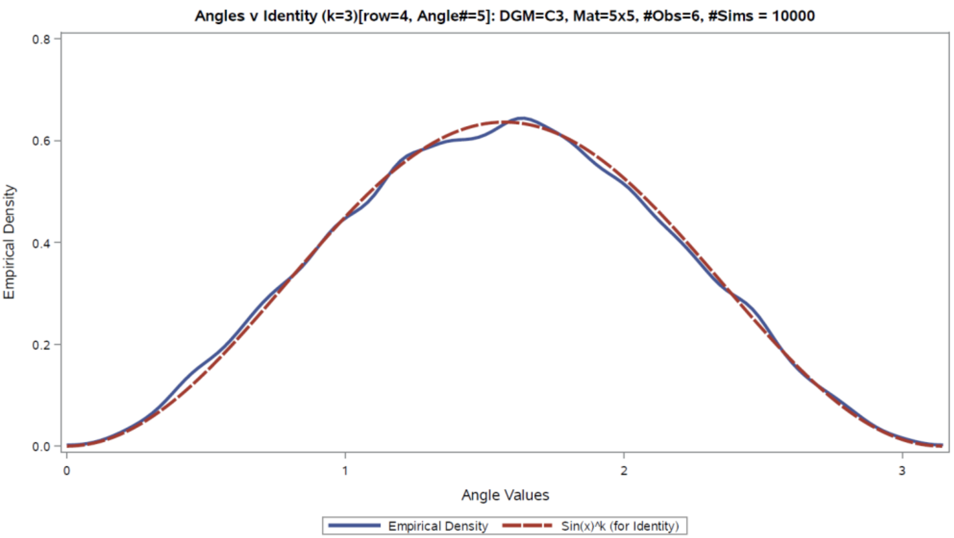
Angle Distributions

C3 (Analytic) **Sample Size n = 6** **MVG Data**



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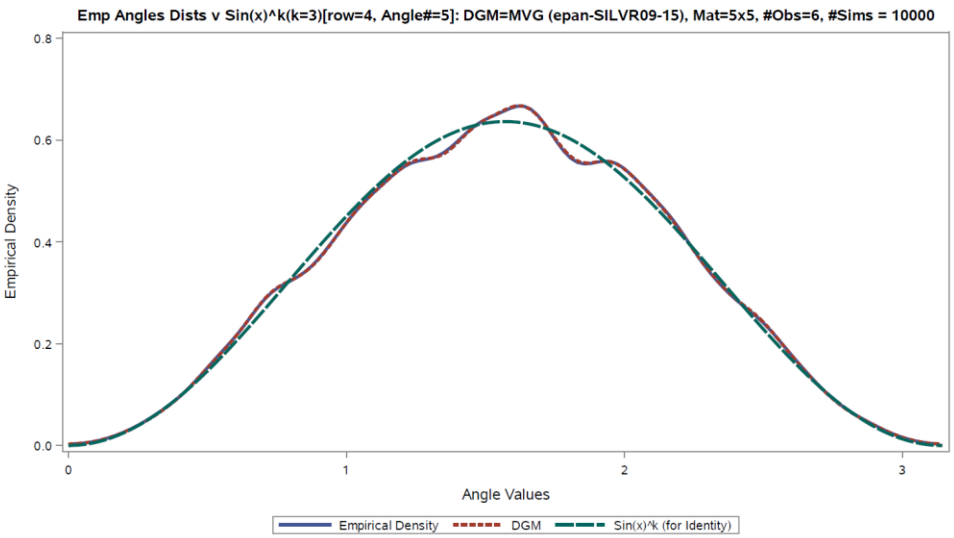

PRNG Seed = 12345
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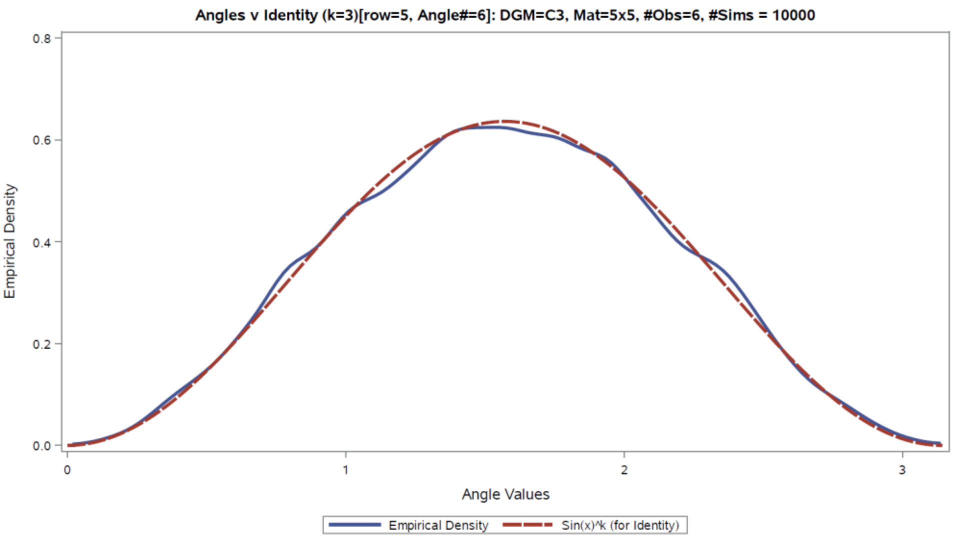
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) Sample Size n = 6 MVG Data



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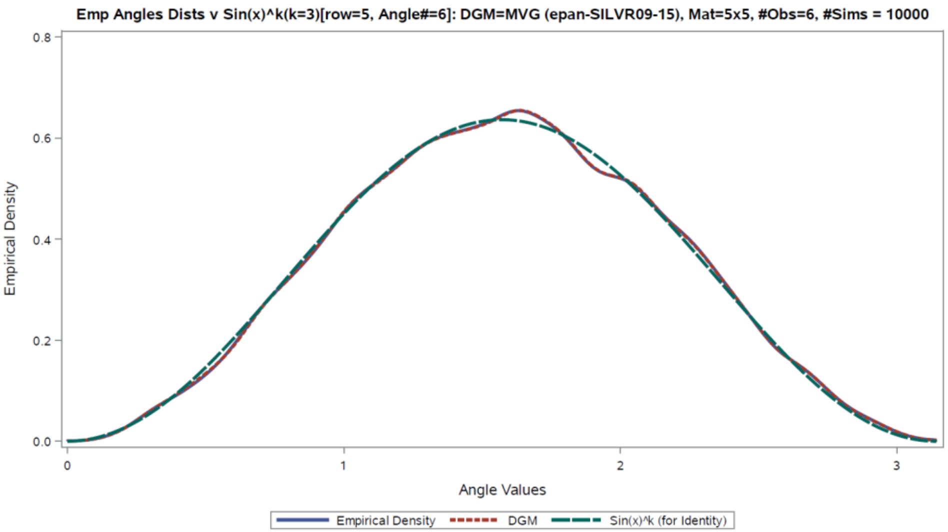
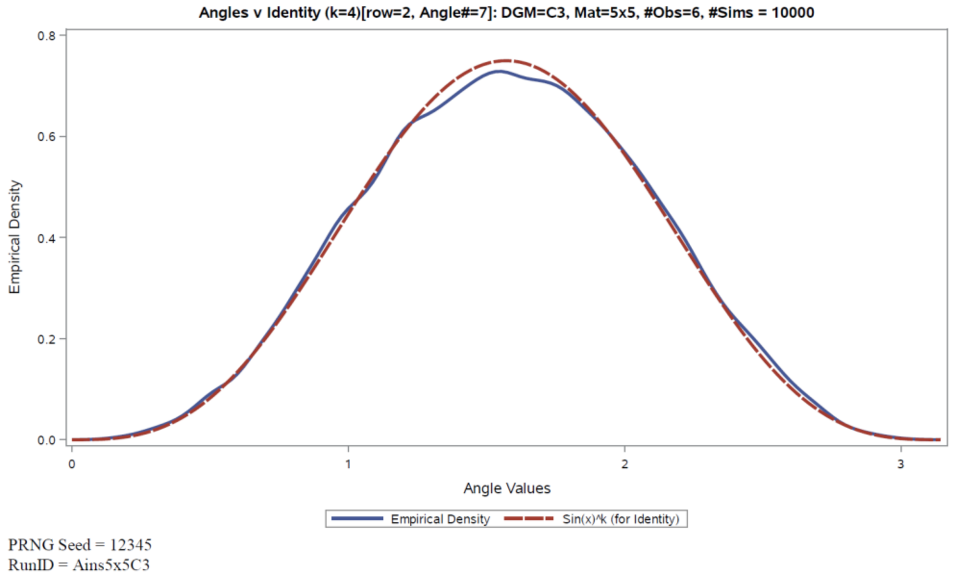


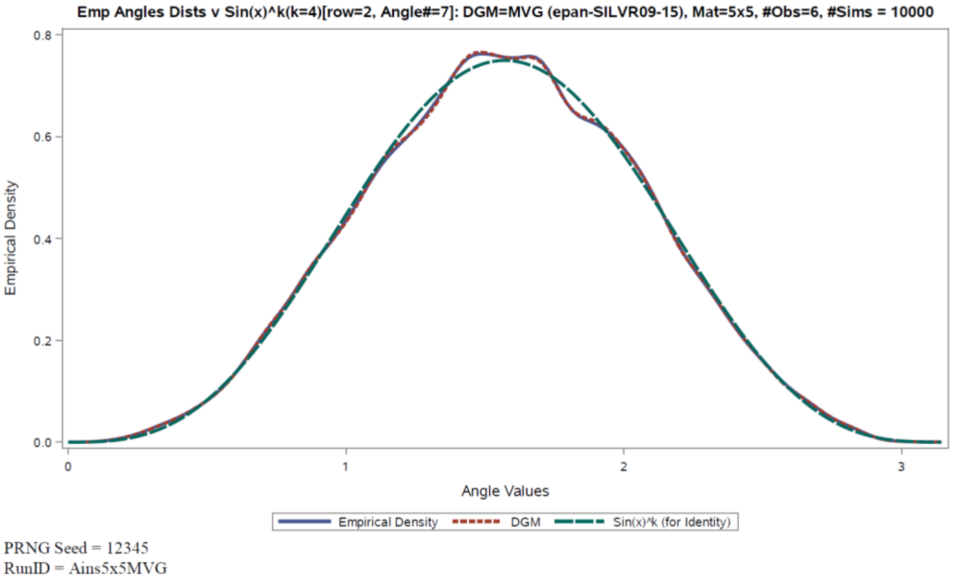
PRNG Seed = 12345
RunID = Ains5x5MVG

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

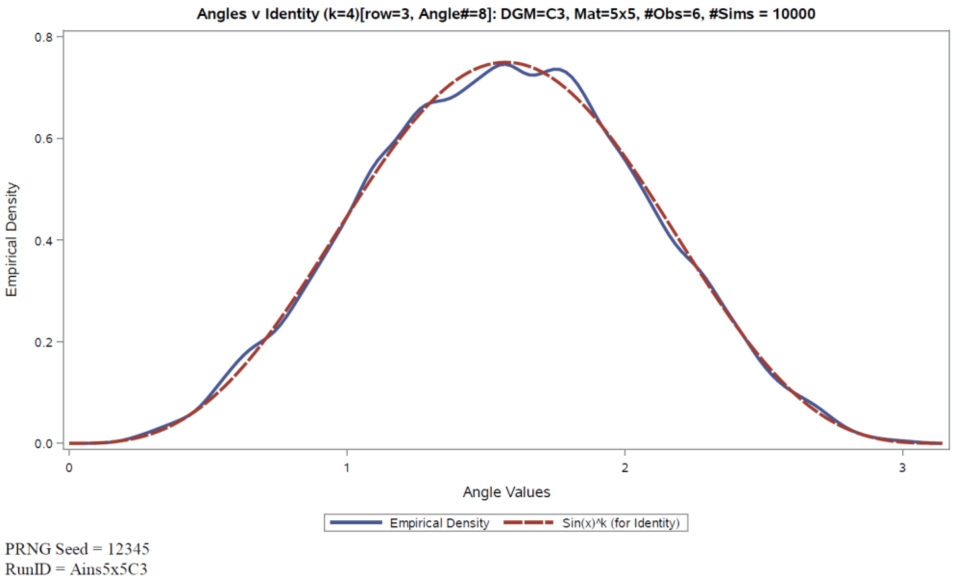
C3 (Analytic) **Sample Size n = 6** **MVG Data**

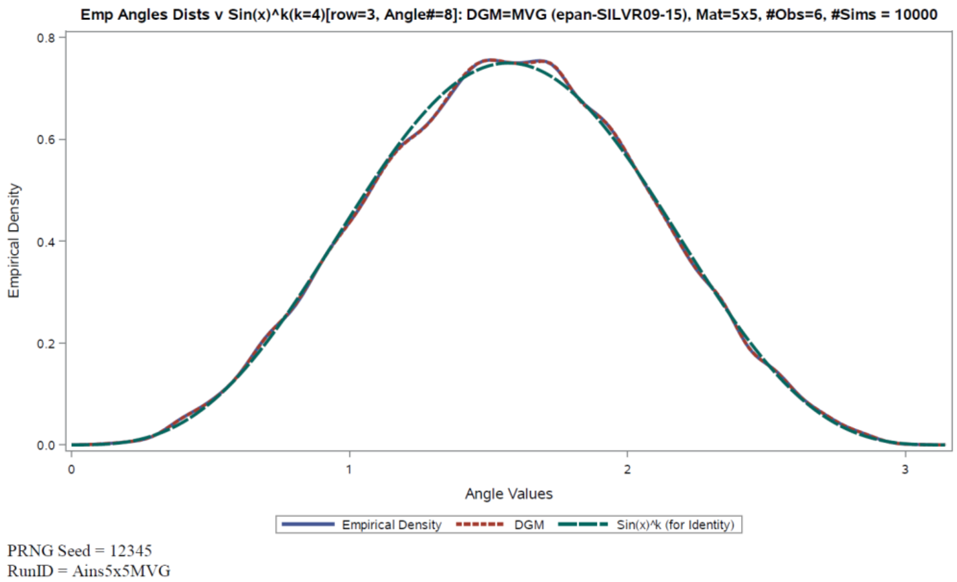




Angle Distributions

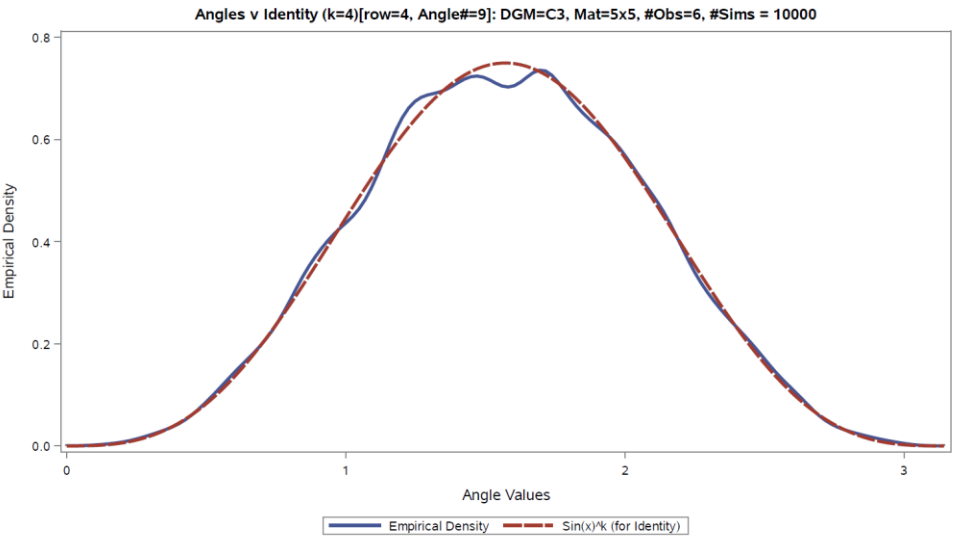
C3 (Analytic) **Sample Size n = 6** **MVG Data**





Angle Distributions

C3 (Analytic) | **Sample Size n = 6** | **MVG Data**



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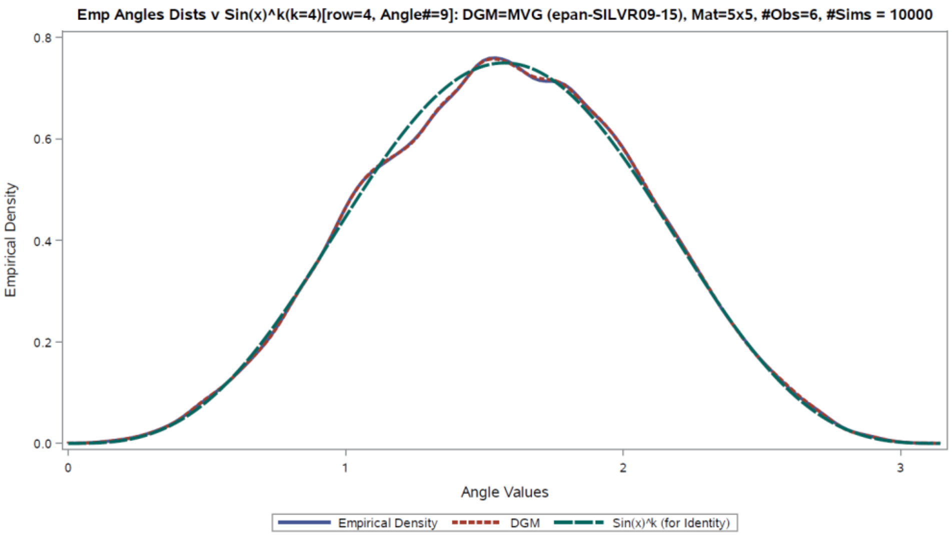

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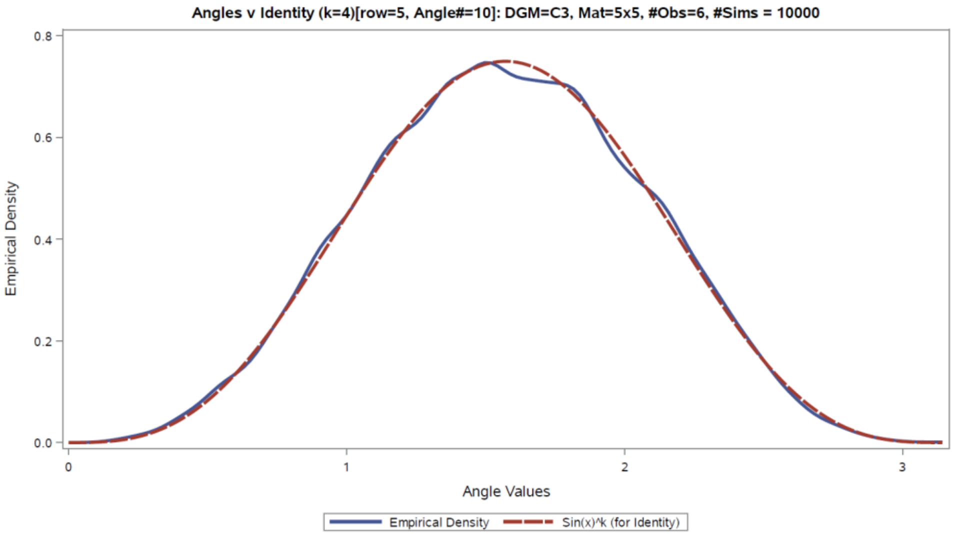
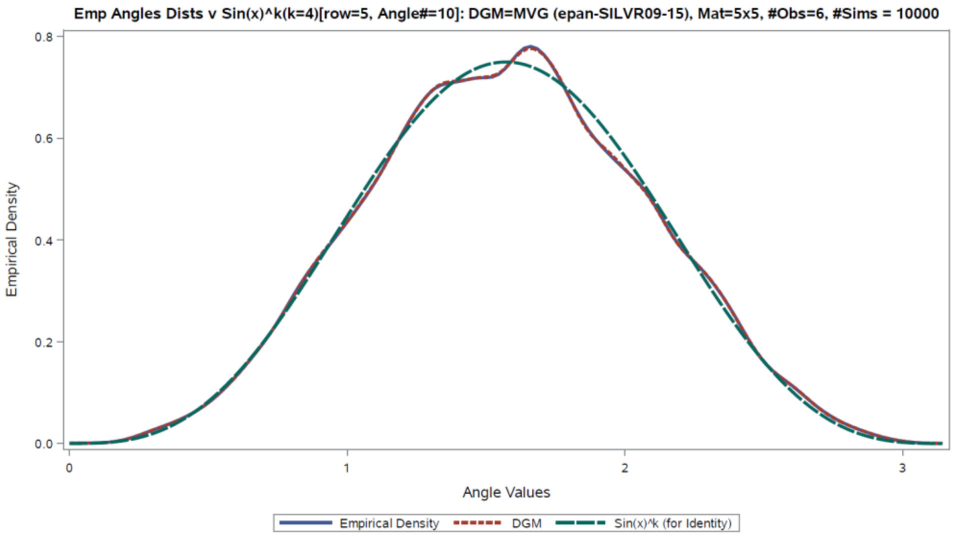
RunID = Ains5x5MVG

Angle Distributions

C3 (Analytic) **Sample Size n = 6** **MVG Data**



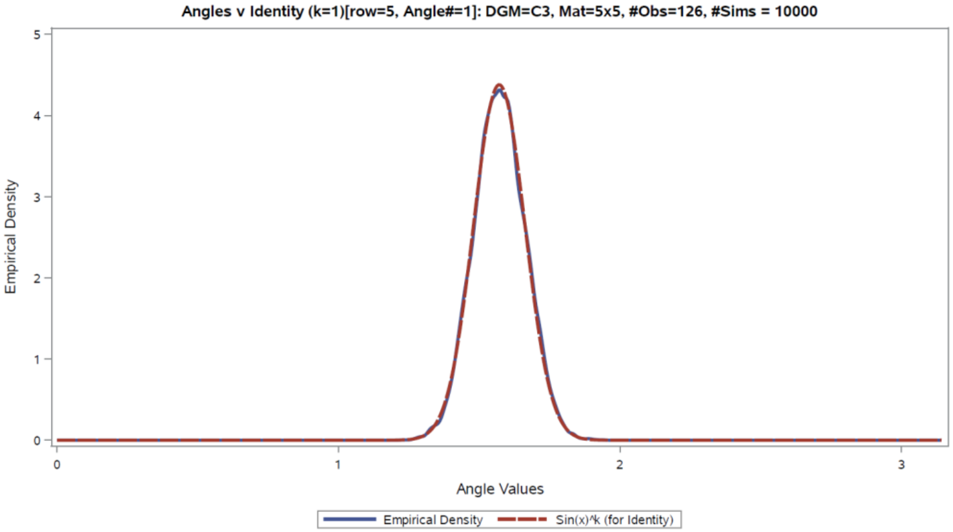
PRNG Seed = 12345
RunID = Ains5x5C3



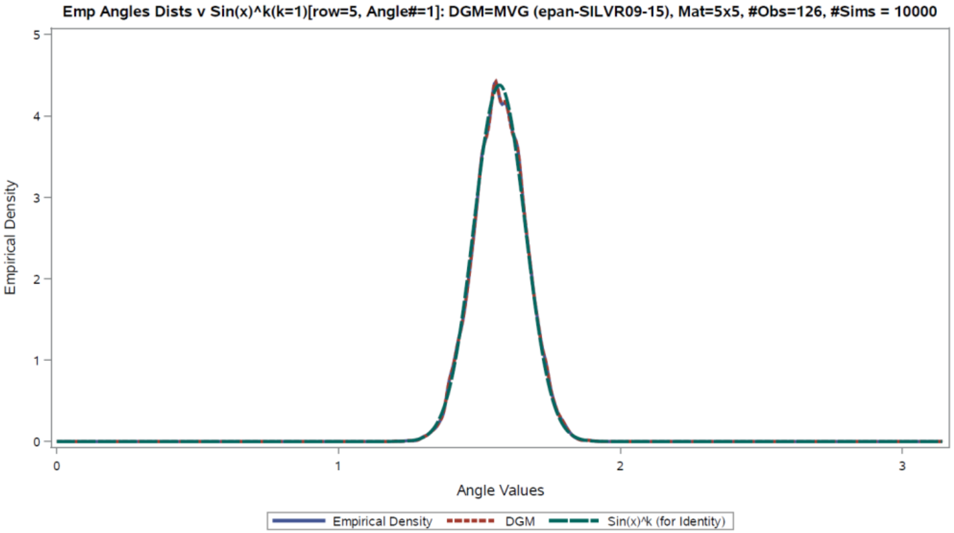
PRNG Seed = 12345
RunID = Ains5x5MVG

Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3

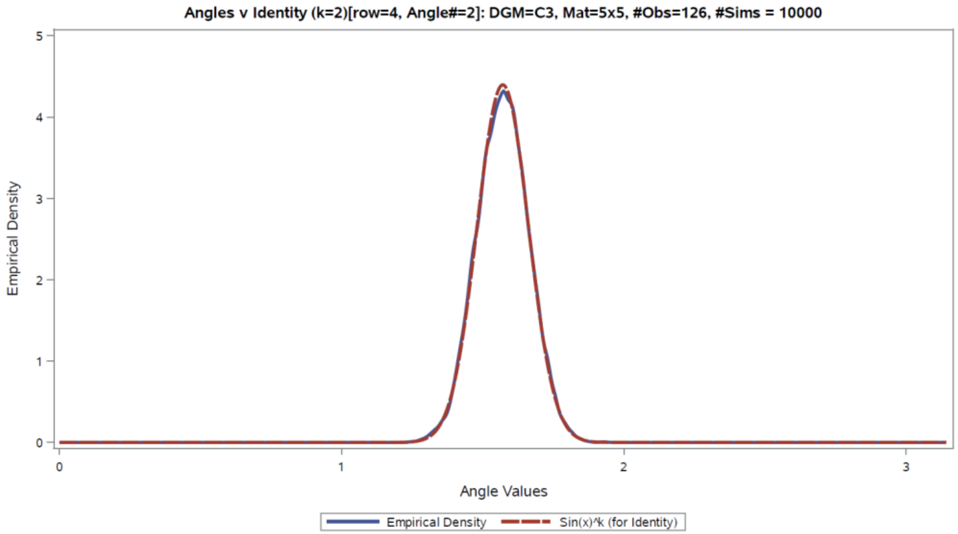


PRNG Seed = 12345
RunID = Ainb5x5MVG

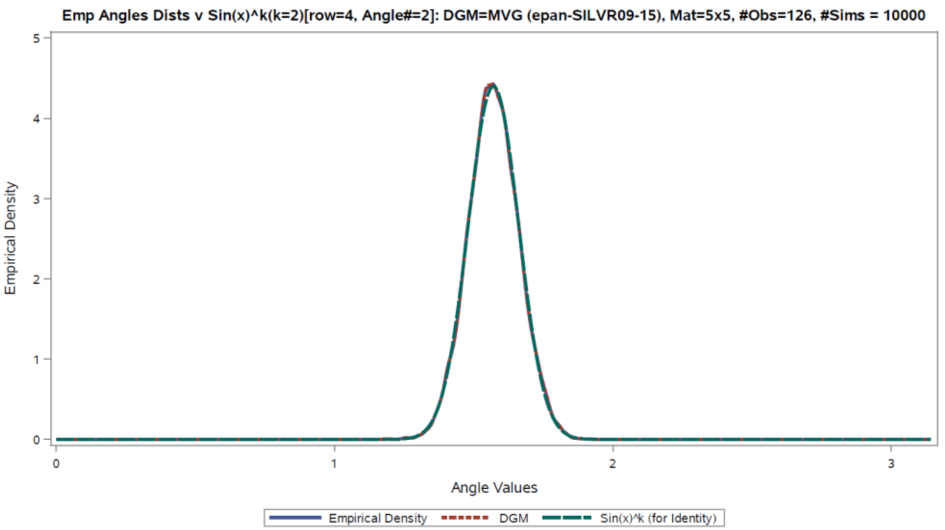
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3

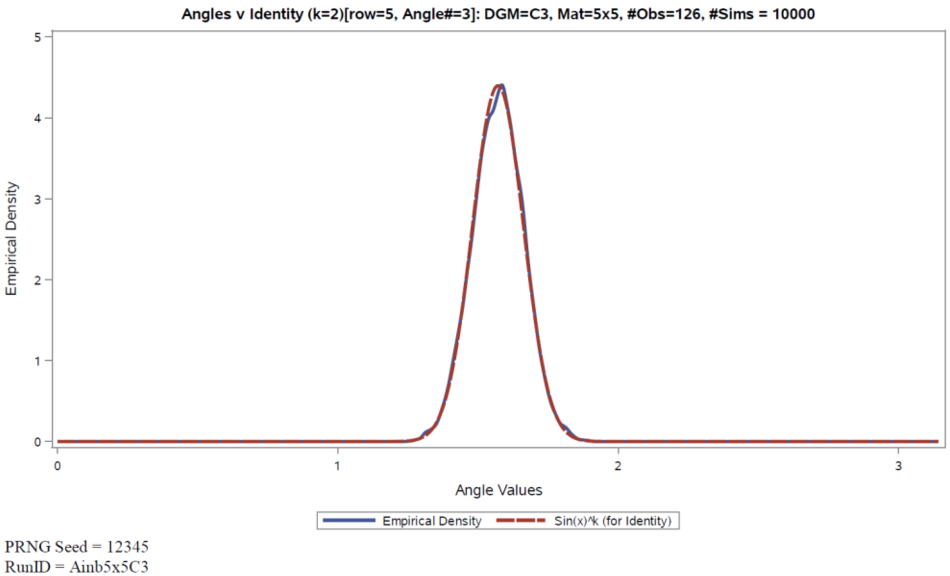


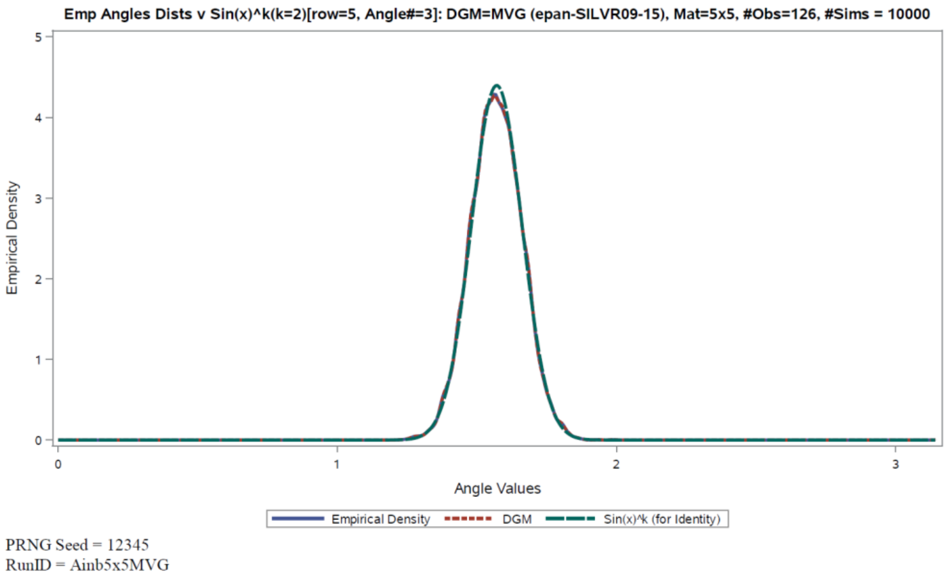
PRNG Seed = 12345
RunID = Ainb5x5MVG

RiskFuse produced by RiskMinds International

Angle Distributions

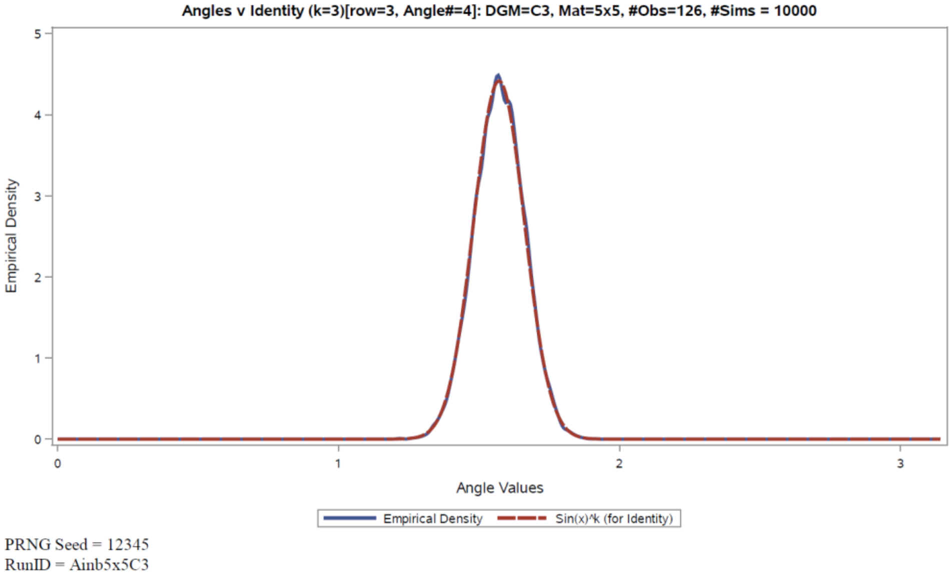
C3 (Analytic) **Sample Size n = 126** **MVG Data**

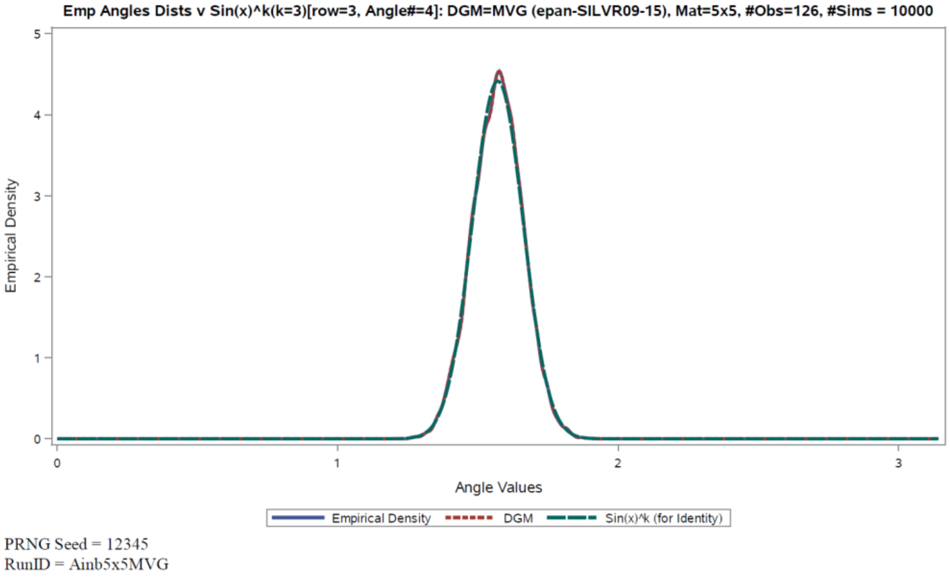




Angle Distributions

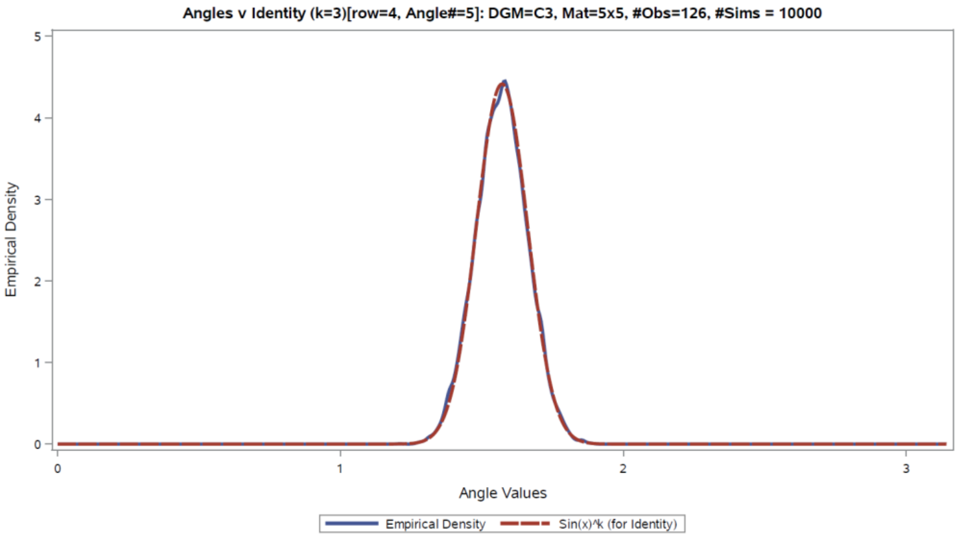
C3 (Analytic) **Sample Size n = 126** **MVG Data**



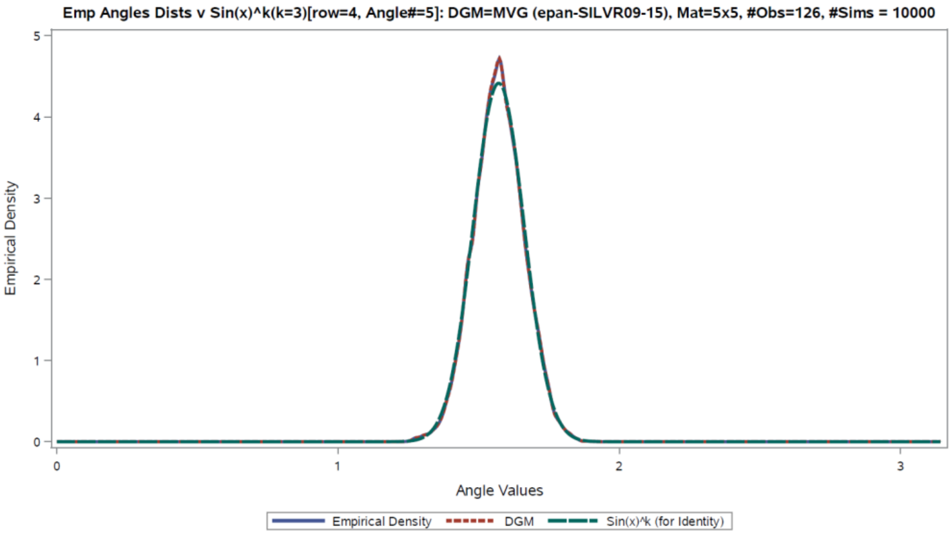


Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3

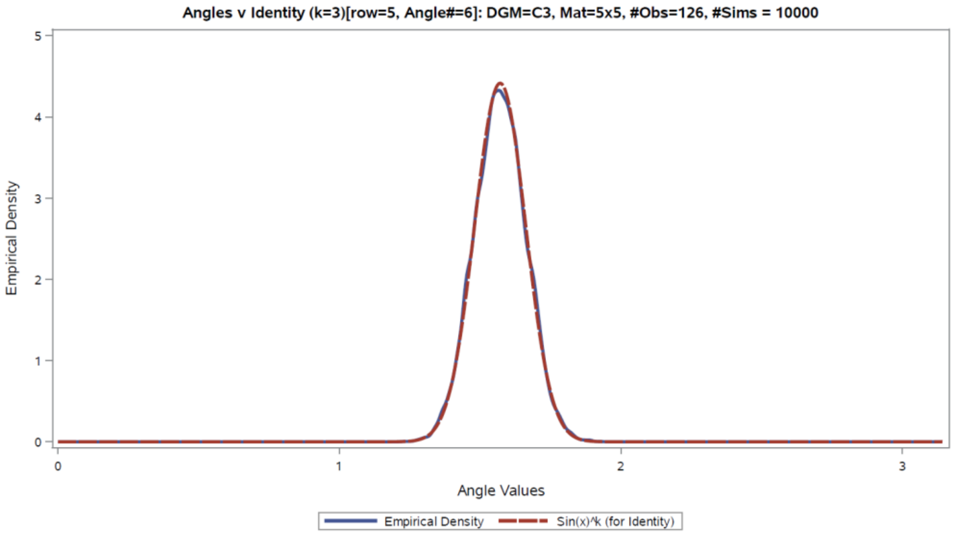


PRNG Seed = 12345
RunID = Ainb5x5MVG

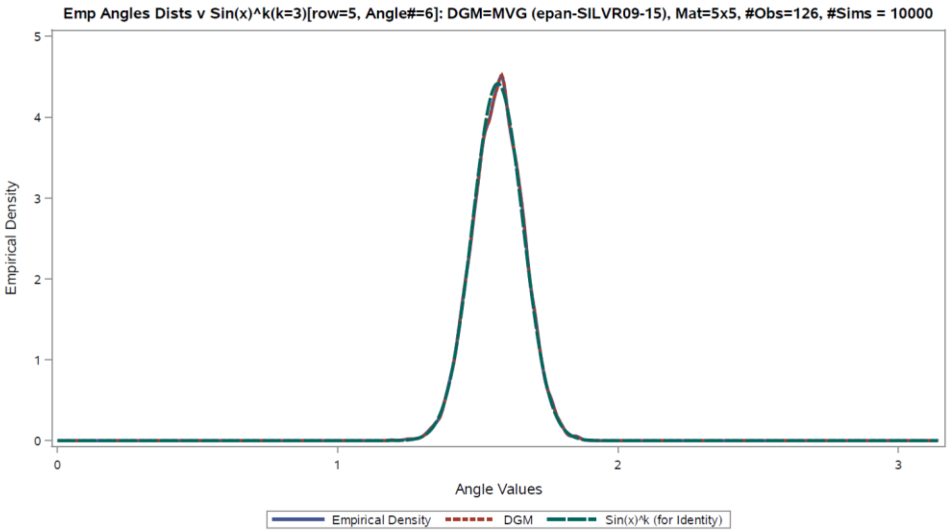
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3

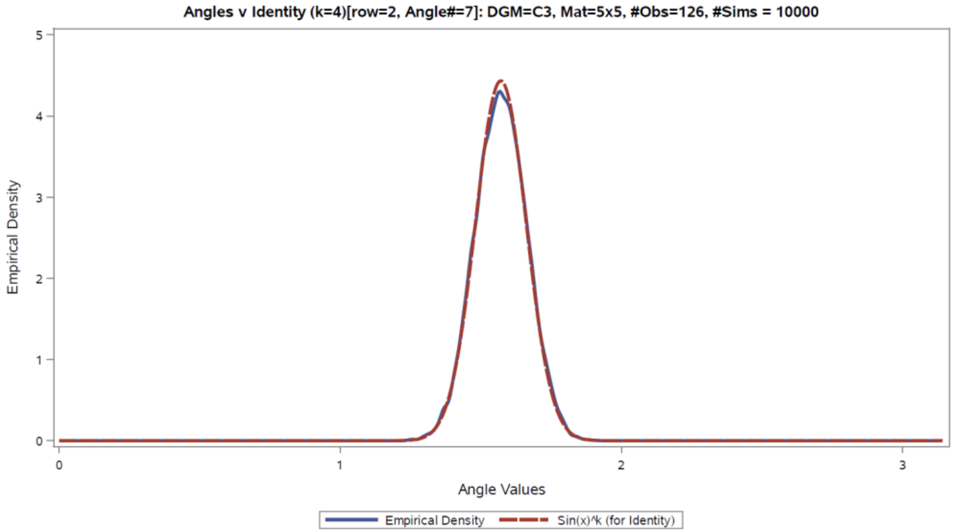


PRNG Seed = 12345
RunID = Ainb5x5MVG

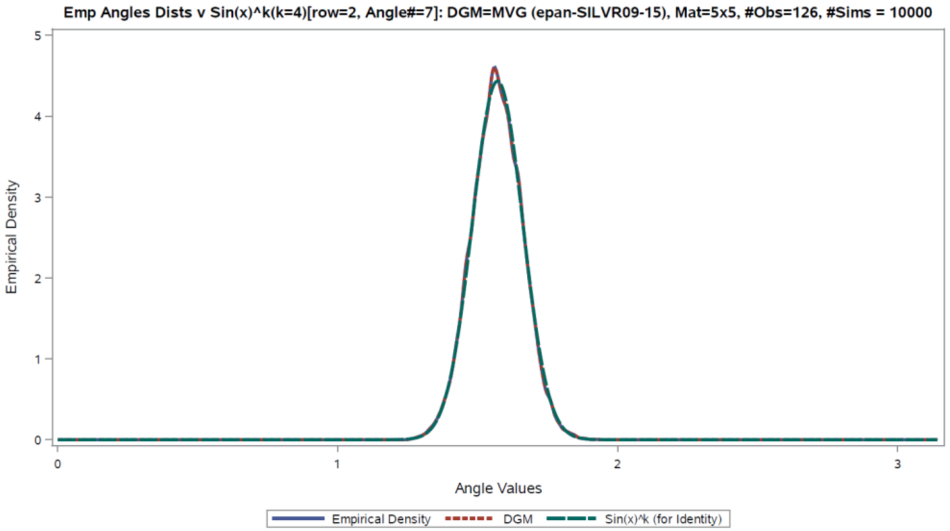
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3

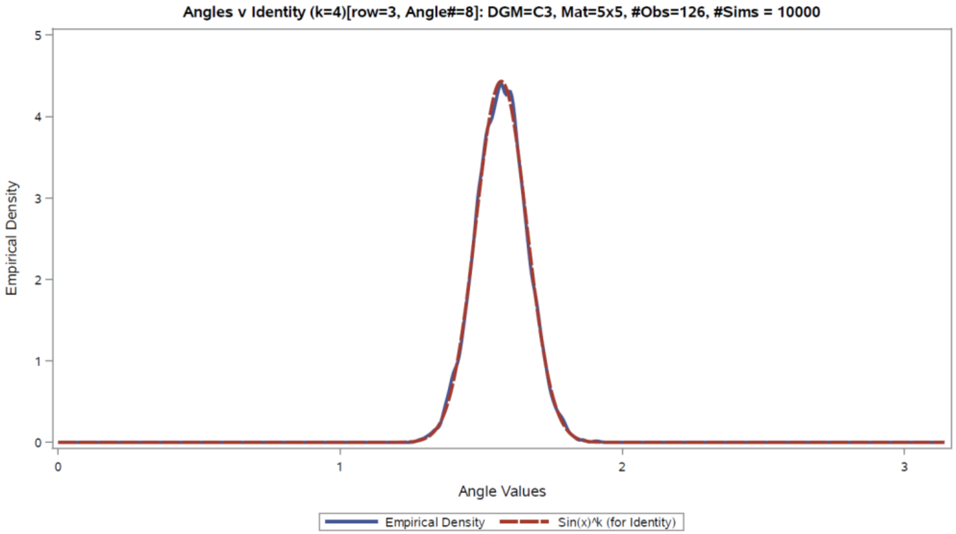


PRNG Seed = 12345
RunID = Ainb5x5MVG

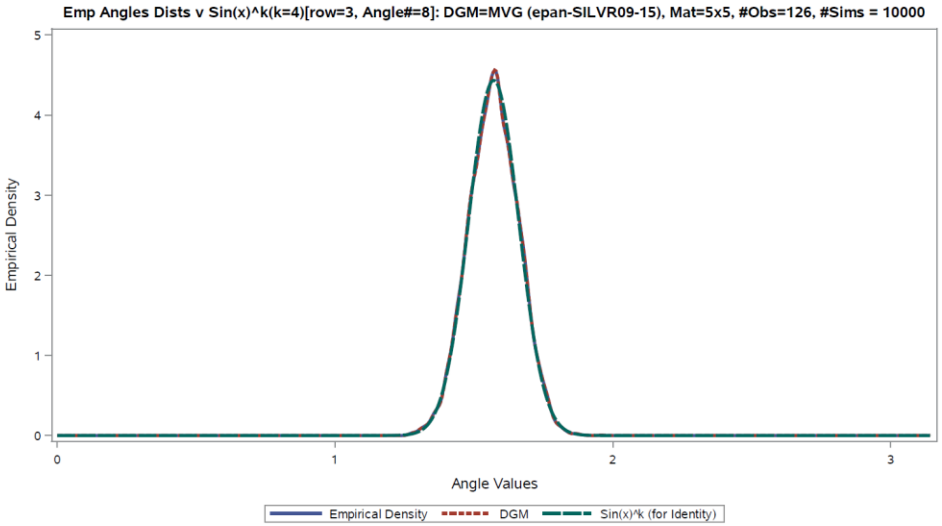
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3

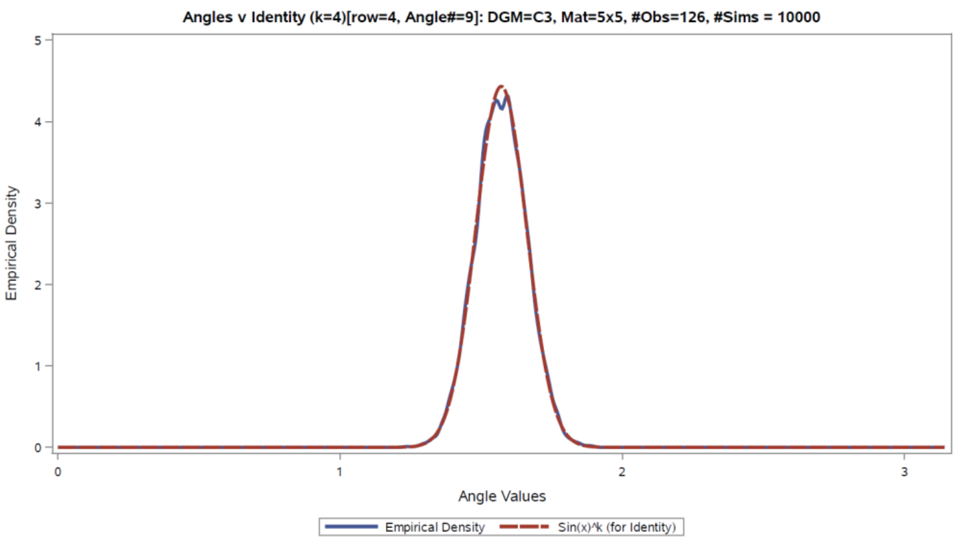


PRNG Seed = 12345
RunID = Ainb5x5MVG

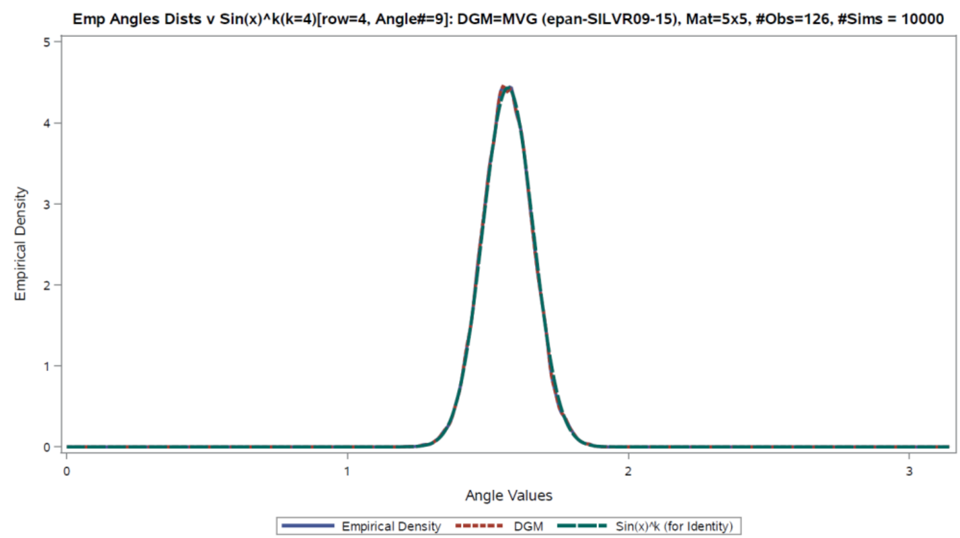
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

Angle Distributions

C3 (Analytic) **Sample Size n = 126** **MVG Data**



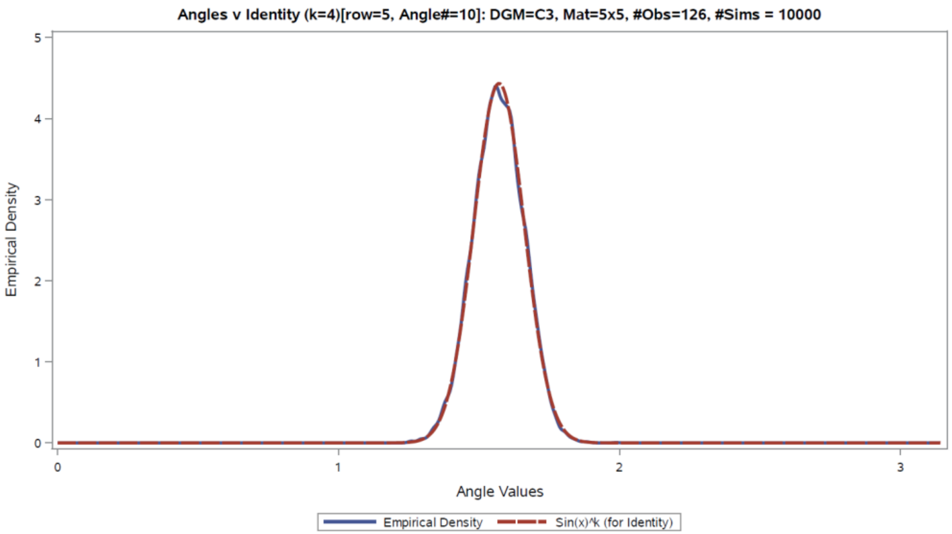
PRNG Seed = 12345
RunID = Ainb5x5C3



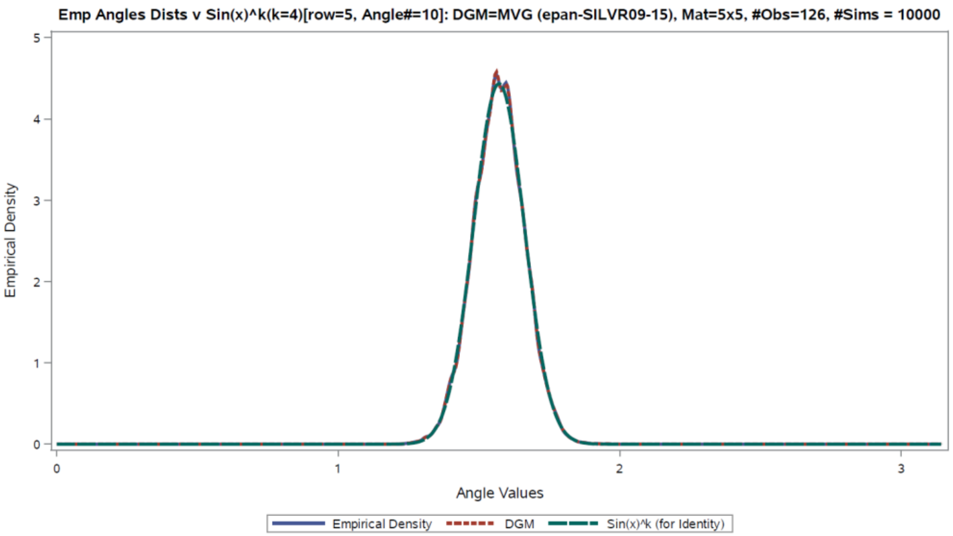
PRNG Seed = 12345
RunID = Ainb5x5MVG

Angle Distributions

C3 (Analytic) — **Sample Size n = 126** — **MVG Data**



PRNG Seed = 12345
RunID = Ainb5x5C3



PRNG Seed = 12345
RunID = Ainb5x5MVG

Correlation Matrices to CDF Matrices (n=6)

Correlation Matrices | **CDFs: C3 (Analytic)** | **CDFs: MVG Data**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.43	0.35	0.57	0.57		0.50	0.43	0.35	0.58	0.58
0	0.1	0.2	-0.1	-0.1		0.50	0.43	0.35	0.57	0.57		0.50	0.42	0.35	0.57	0.57
0	0.1	0.2	-0.1	-0.1		0.50	0.43	0.35	0.57	0.57		0.49	0.42	0.35	0.57	0.57
0	0.1	0.2	-0.1	-0.1		0.50	0.43	0.35	0.57	0.57		0.50	0.42	0.35	0.58	0.58
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.50	0.44	0.39	0.57	0.38		0.50	0.44	0.40	0.57	0.38
0	0.1	0.2	-0.1	0.2		0.50	0.44	0.39	0.57	0.38		0.49	0.44	0.39	0.57	0.37
0	0.1	0.2	-0.1	0.2		0.50	0.44	0.39	0.57	0.38		0.49	0.44	0.39	0.57	0.37
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.46	0.43	0.56	0.58		0.50	0.46	0.43	0.56	0.58
0	0.1	0.2	-0.1	-0.1		0.50	0.46	0.43	0.56	0.58		0.50	0.46	0.43	0.56	0.57
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.50	0.48	0.46	0.55	0.42		0.50	0.48	0.47	0.55	0.43
1	1	1	1	1												

Correlation Matrices to CDF Matrices (n=126)

Correlation Matrices → **CDFs: C3 (Analytic)** **CDFs: MVG Data**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.13	0.01	0.87	0.87		0.51	0.14	0.01	0.87	0.87
0	0.1	0.2	-0.1	-0.1		0.50	0.13	0.01	0.87	0.87		0.50	0.13	0.01	0.86	0.86
0	0.1	0.2	-0.1	-0.1		0.50	0.13	0.01	0.87	0.87		0.51	0.13	0.01	0.87	0.87
0	0.1	0.2	-0.1	-0.1		0.50	0.13	0.01	0.87	0.87		0.50	0.13	0.01	0.87	0.87
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.50	0.16	0.03	0.89	0.02		0.50	0.16	0.03	0.90	0.02
0	0.1	0.2	-0.1	0.2		0.50	0.16	0.03	0.89	0.02		0.50	0.15	0.03	0.89	0.01
0	0.1	0.2	-0.1	0.2		0.50	0.16	0.03	0.89	0.02		0.49	0.16	0.03	0.89	0.02
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.18	0.06	0.92	0.96		0.50	0.18	0.06	0.91	0.95
0	0.1	0.2	-0.1	-0.1		0.50	0.18	0.06	0.92	0.96		0.51	0.19	0.06	0.92	0.96
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.50	0.20	0.08	0.94	0.00		0.50	0.19	0.08	0.94	0.00
1	1	1	1	1												

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

CDF Matrices to Correlation Matrices (n=6)

CDF Matrices | **Correlations: C3 (Analytic)** | **Correlations: MVG Data**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.00	0.13	-0.13	0.20	-0.61		0.00	0.13	-0.13	0.20	-0.61
0.5	0.4	0.6	0.35	0.9		0.00	0.13	-0.13	0.20	-0.61		0.00	0.13	-0.14	0.19	-0.60
0.5	0.4	0.6	0.35	0.9		0.00	0.13	-0.13	0.20	-0.61		-0.01	0.12	-0.14	0.20	-0.61
0.5	0.4	0.6	0.35	0.9		0.00	0.13	-0.13	0.20	-0.61		0.00	0.14	-0.13	0.21	-0.62
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.00	0.17	-0.14	-0.11	0.06		0.00	0.18	-0.14	-0.11	0.05
0.5	0.4	0.6	0.6	0.8		0.00	0.17	-0.14	-0.11	0.06		-0.01	0.16	-0.13	-0.11	0.06
0.5	0.4	0.6	0.6	0.8		0.00	0.17	-0.14	-0.11	0.06		-0.01	0.16	-0.14	-0.12	0.07
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.00	0.23	-0.15	0.25	0.71		0.00	0.24	-0.14	0.26	0.72
0.5	0.4	0.6	0.4	0.3		0.00	0.23	-0.15	0.25	0.71		0.00	0.23	-0.16	0.25	0.72
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.00	0.36	-0.20	-0.04	0.20		0.02	0.38	-0.17	-0.02	0.23
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

CDF Matrices to Correlation Matrices (n=126)

CDF Matrices | **Correlations: C3 (Analytic)** | **Correlations: MVG Data**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.00	0.02	-0.02	0.03	-0.11		0.00	0.02	-0.02	0.04	-0.11
0.5	0.4	0.6	0.35	0.9		0.00	0.02	-0.02	0.03	-0.11		0.00	0.02	-0.02	0.03	-0.12
0.5	0.4	0.6	0.35	0.9		0.00	0.02	-0.02	0.03	-0.11		0.00	0.02	-0.02	0.04	-0.11
0.5	0.4	0.6	0.35	0.9		0.00	0.02	-0.02	0.03	-0.11		0.00	0.02	-0.02	0.03	-0.11
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.00	0.02	-0.02	-0.02	-0.06		0.00	0.02	-0.02	-0.02	-0.06
0.5	0.4	0.6	0.6	0.8		0.00	0.02	-0.02	-0.02	-0.06		0.00	0.02	-0.02	-0.02	-0.06
0.5	0.4	0.6	0.6	0.8		0.00	0.02	-0.02	-0.02	-0.06		0.00	0.02	-0.02	-0.02	-0.06
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.00	0.02	-0.02	0.02	0.07		0.00	0.02	-0.02	0.03	0.06
0.5	0.4	0.6	0.4	0.3		0.00	0.02	-0.02	0.02	0.07		0.00	0.03	-0.02	0.03	0.07
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.00	0.02	-0.02	-0.01	-0.12		0.00	0.02	-0.02	-0.01	-0.13
						1	1	1	1	1		1	1	1	1	1

CDF %Shift Matrices to Correlation Matrices (n=6)

CDF %Shift Matrices | **Correlations: C3 (Analytic)** | **Correlations: MVG Data**

					lnMatPr	-2.231	-2.231	-6.931	-6.931	-12.799		-2.222	-2.241	-6.893	-6.970	-12.755
					FNorm	0.899	0.654	2.308	1.160	3.040		0.907	0.638	2.322	1.160	3.045
					Rnk_lnMat	1	1	3	3	5		1	2	3	4	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.13	-0.13	0.35	-0.35	0.43		0.13	-0.13	0.35	-0.35	0.43
20	-20	50	-50	60		0.13	-0.13	0.35	-0.35	0.43		0.13	-0.14	0.35	-0.34	0.42
20	-20	50	-50	60		0.13	-0.13	0.35	-0.35	0.43		0.12	-0.14	0.35	-0.35	0.43
20	-20	50	-50	60		0.13	-0.13	0.35	-0.35	0.43		0.14	-0.13	0.35	-0.35	0.42
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.17	-0.14	0.48	-0.23	0.66		0.18	-0.13	0.48	-0.24	0.67
20	-20	50	-50	70		0.17	-0.14	0.48	-0.23	0.66		0.17	-0.13	0.48	-0.24	0.66
20	-20	50	-50	70		0.17	-0.14	0.48	-0.23	0.66		0.17	-0.14	0.47	-0.24	0.65
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.23	-0.15	0.63	-0.10	0.89		0.24	-0.14	0.64	-0.10	0.90
20	-20	50	-50	80		0.23	-0.15	0.63	-0.10	0.89		0.24	-0.15	0.64	-0.10	0.90
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.36	-0.20	0.84	0.06	1.00		0.37	-0.18	0.84	0.07	1.00
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case A

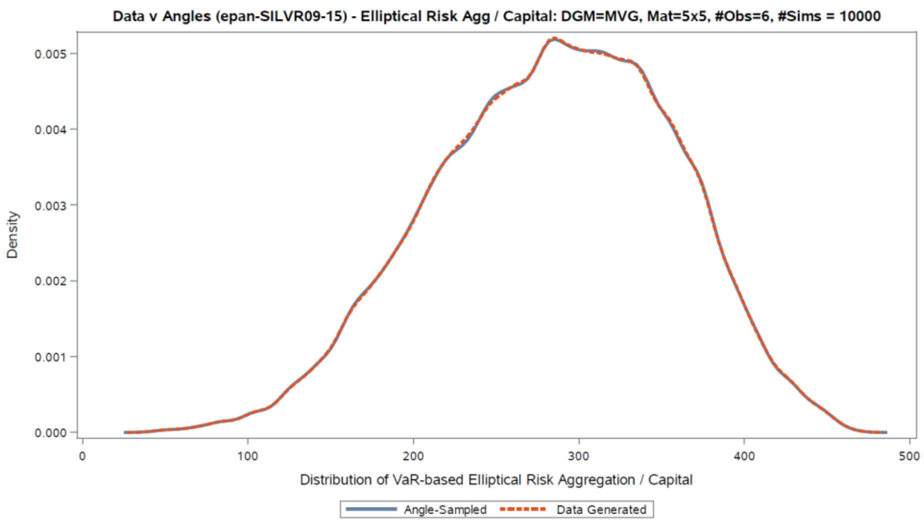
CDF %Shift Matrices to Correlation Matrices (n=126)

CDF %Shift Matrices | **Correlations: C3 (Analytic)** | **Correlations: MVG Data**

					lnMatPr	-2.231	-2.231	-6.931	-6.931	-12.799		-2.233	-2.230	-6.938	-6.925	-12.881
					FNorm	0.105	0.100	0.288	0.255	0.479		0.103	0.099	0.286	0.255	0.480
					Rnk_lnMat	1	1	3	3	5		2	1	4	3	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.02	-0.02	0.06	-0.06	0.08		0.02	-0.02	0.06	-0.06	0.08
20	-20	50	-50	60		0.02	-0.02	0.06	-0.06	0.08		0.02	-0.02	0.06	-0.06	0.08
20	-20	50	-50	60		0.02	-0.02	0.06	-0.06	0.08		0.02	-0.02	0.06	-0.06	0.08
20	-20	50	-50	60		0.02	-0.02	0.06	-0.06	0.08		0.02	-0.02	0.06	-0.06	0.08
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.02	-0.02	0.06	-0.06	0.10		0.02	-0.02	0.06	-0.06	0.10
20	-20	50	-50	70		0.02	-0.02	0.06	-0.06	0.10		0.02	-0.02	0.06	-0.06	0.10
20	-20	50	-50	70		0.02	-0.02	0.06	-0.06	0.10		0.02	-0.02	0.06	-0.06	0.10
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.02	-0.02	0.07	-0.05	0.13		0.02	-0.02	0.07	-0.05	0.13
20	-20	50	-50	80		0.02	-0.02	0.07	-0.05	0.13		0.03	-0.02	0.07	-0.05	0.13
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.02	-0.02	0.07	-0.05	0.17		0.02	-0.02	0.07	-0.05	0.17
						1	1	1	1	1		1	1	1	1	1

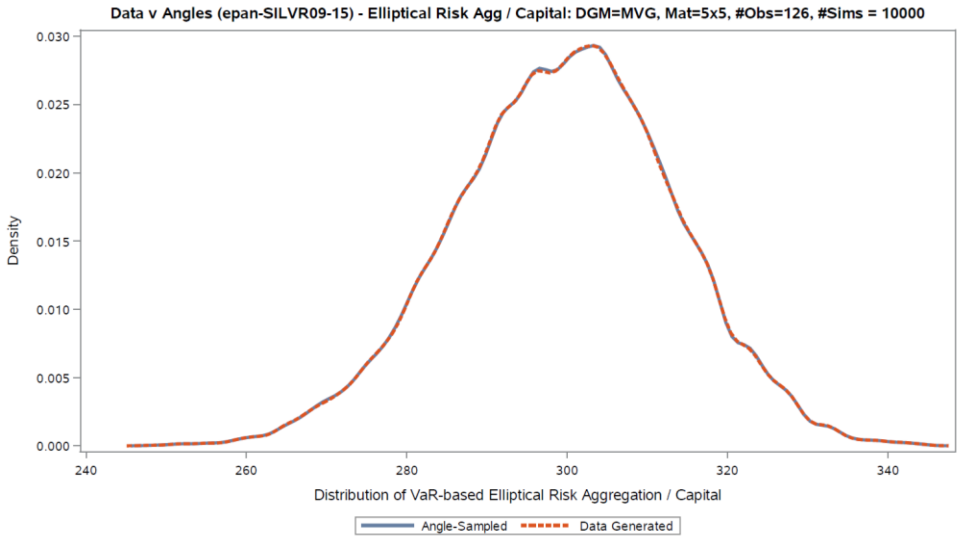
Elliptical Capital

n = 6



PRNG Seed = 12345
RunID = Bns5x5MVG

n = 126

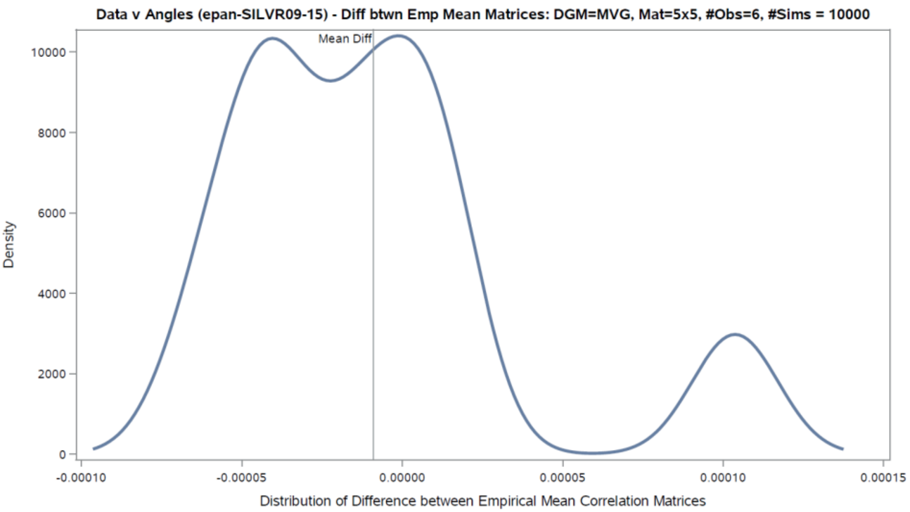


PRNG Seed = 12345
RunID = Bnb5x5MVG

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

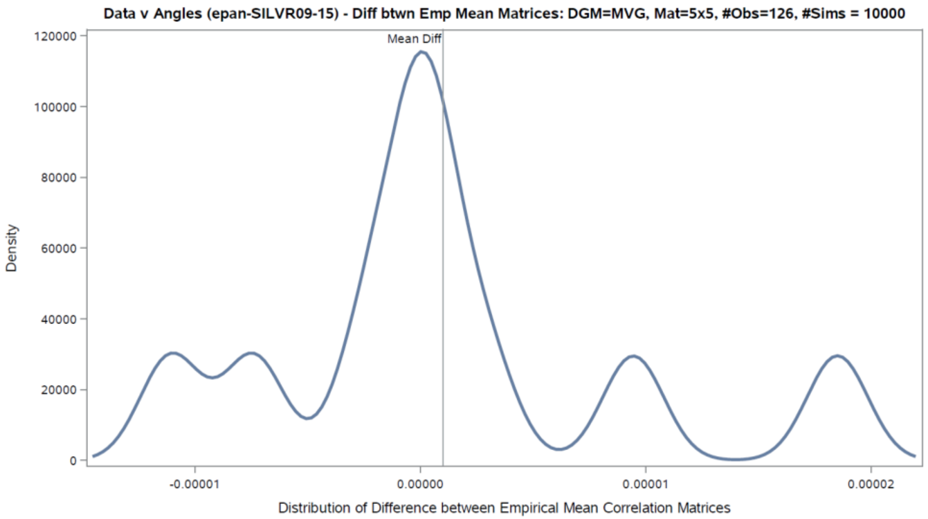
Difference Between Mean Empirical Matrices

n = 6



PRNG Seed = 12345
RunID = Bns5x5MVG

n = 126

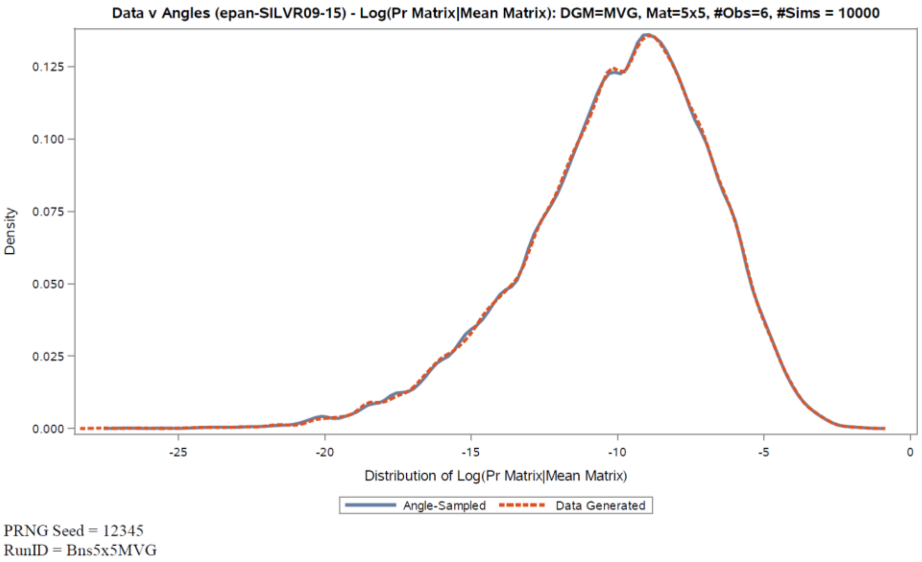


PRNG Seed = 12345
RunID = Bnb5x5MVG

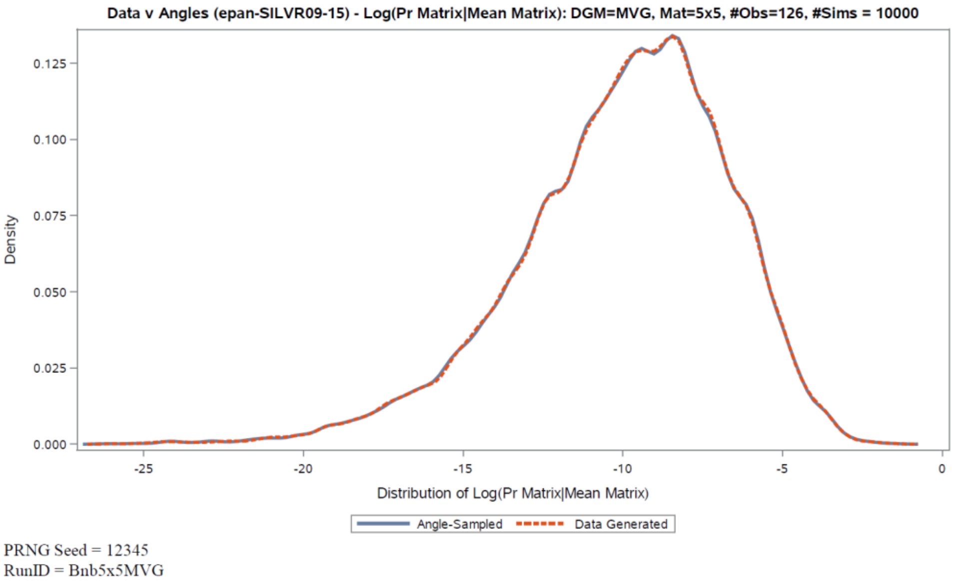
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

LNP

n = 6

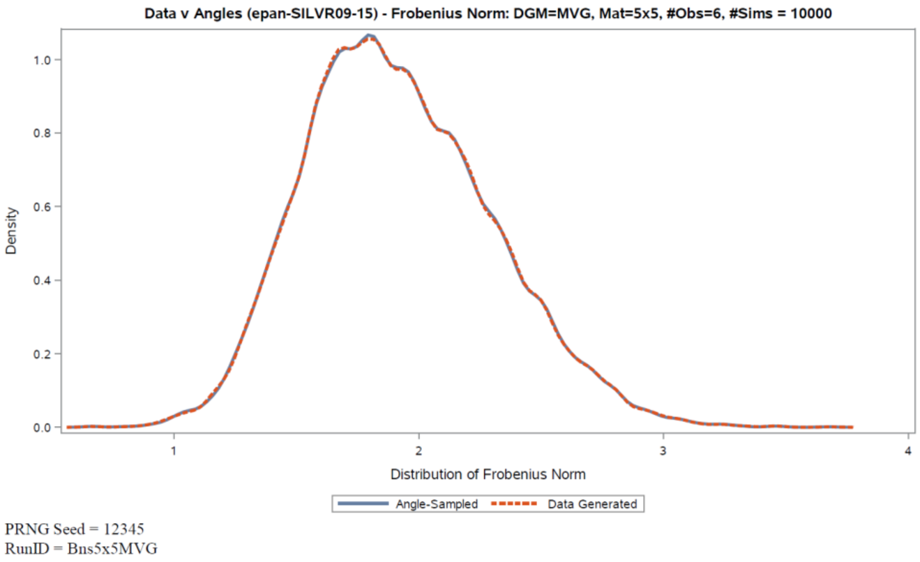


n = 126

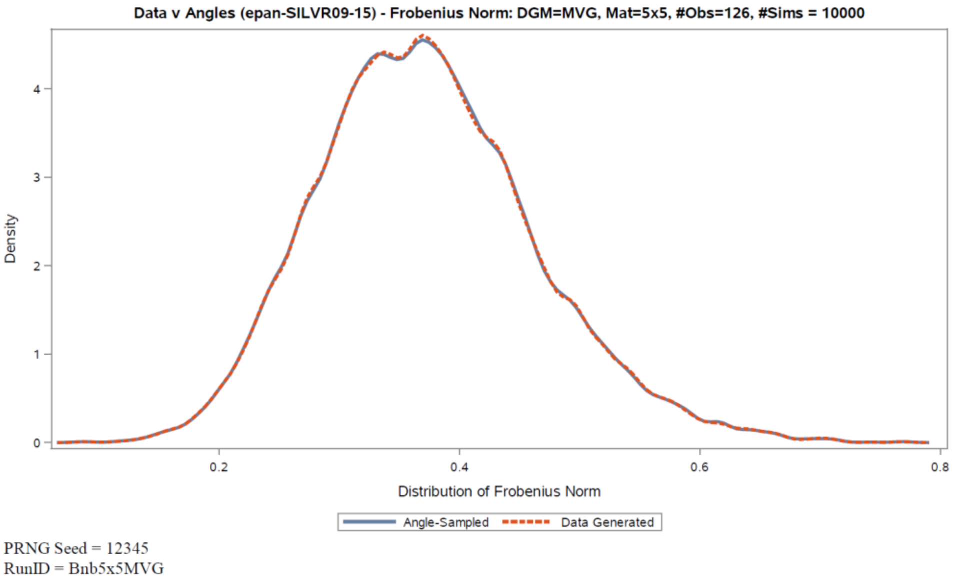


Euclidian/Frobenius Norm

n = 6



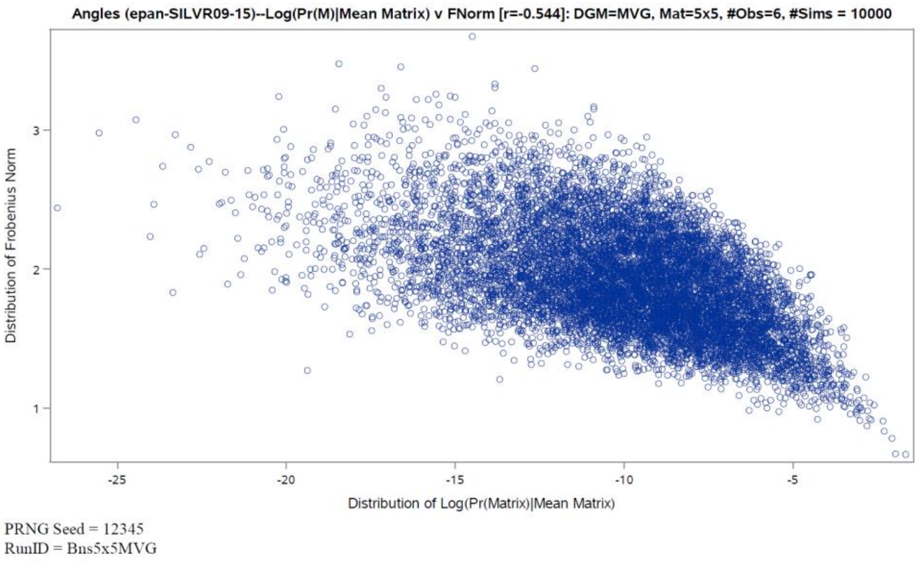
n = 126

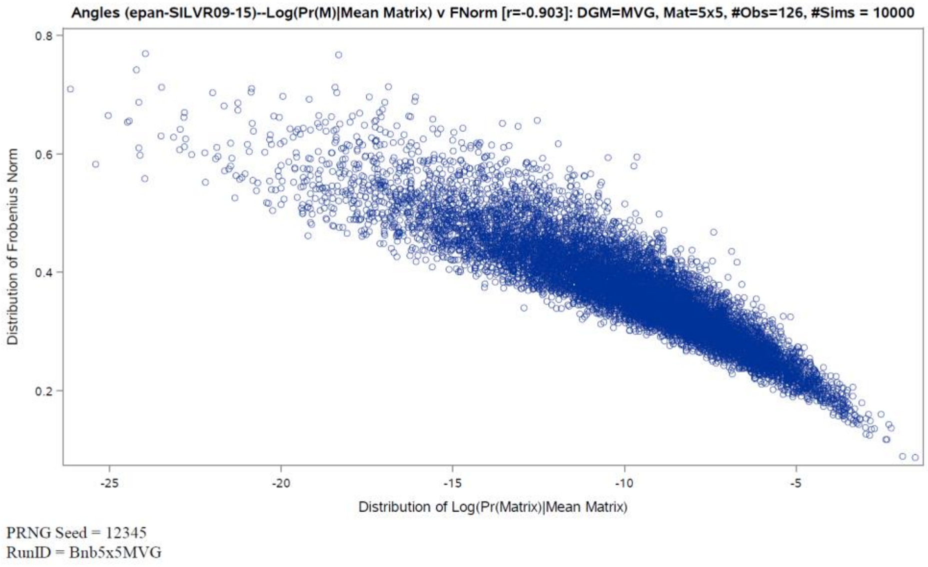


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

LNP v Euclidian/Frobenius Norm

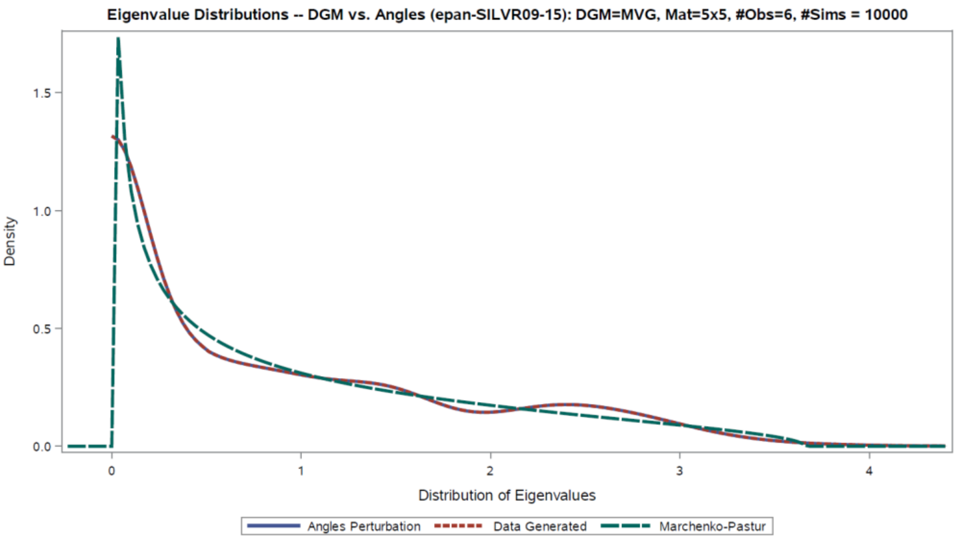
n = 6 | **n = 126**



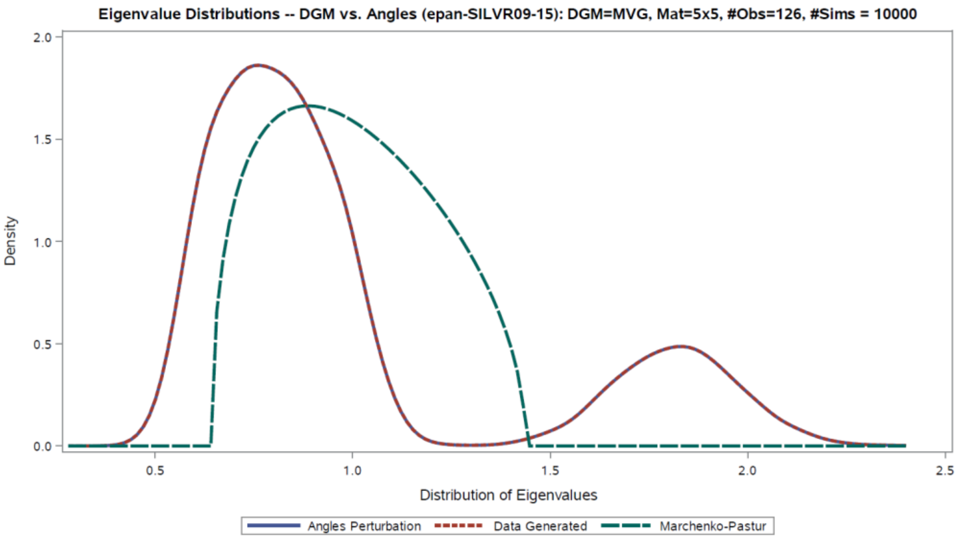


Spectral Distributions

n = 6 | **n = 126**



PRNG Seed = 12345
RunID = Bns5x5MVG

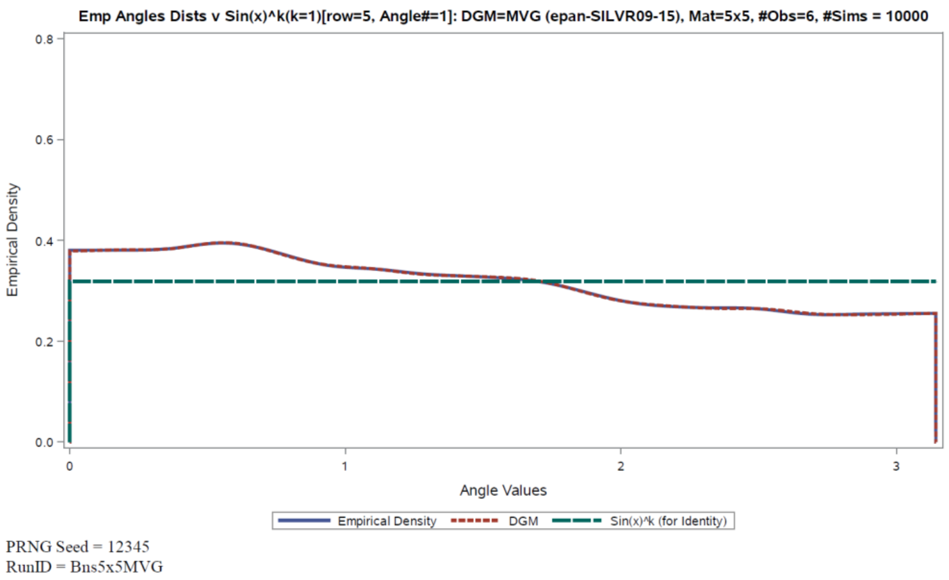


PRNG Seed = 12345
RunID = Bnb5x5MVG

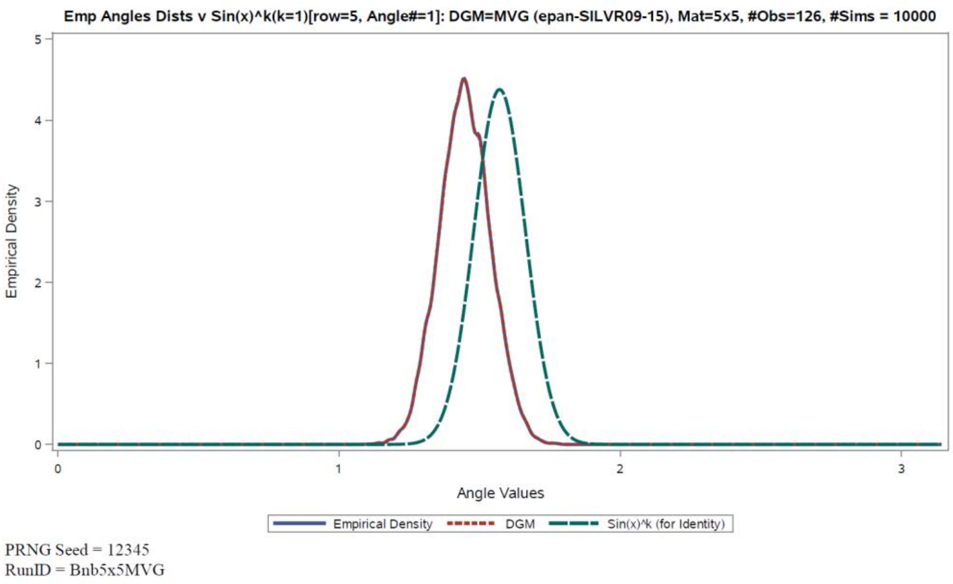
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

Angle Distributions

n = 6



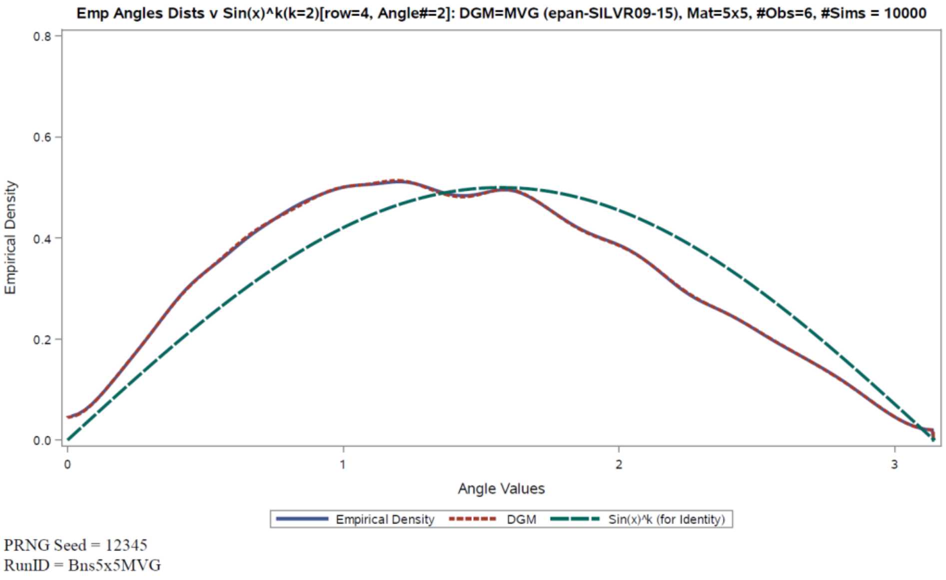
n = 126

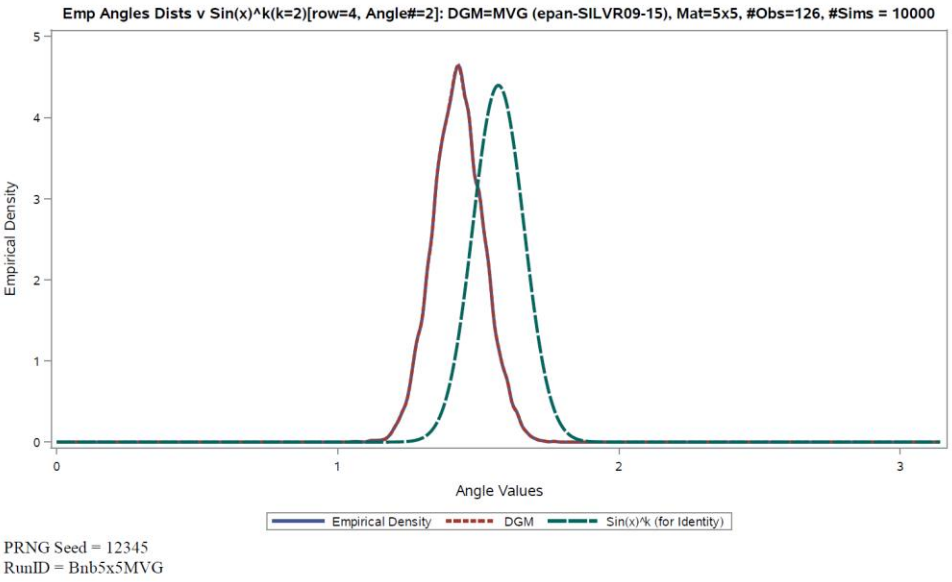


Angle Distributions

n = 6

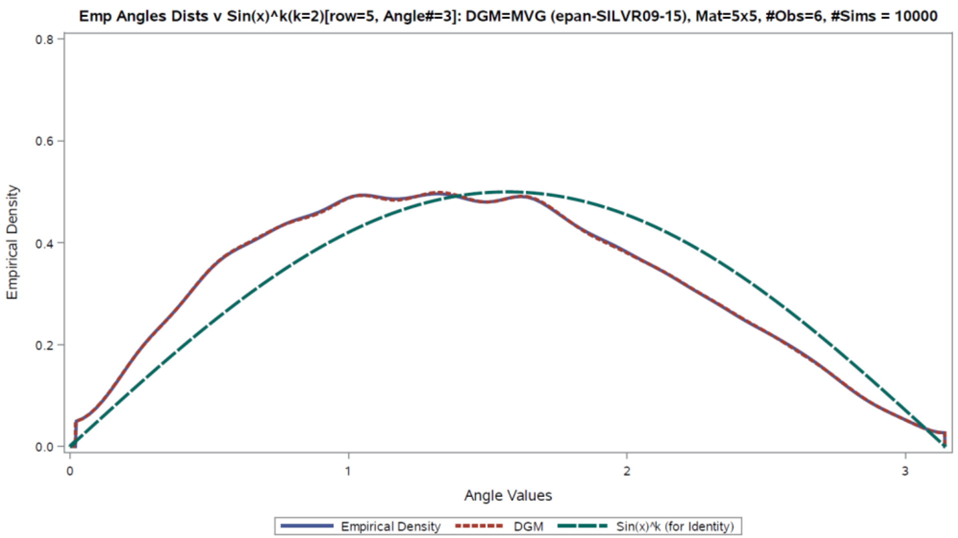
n = 126



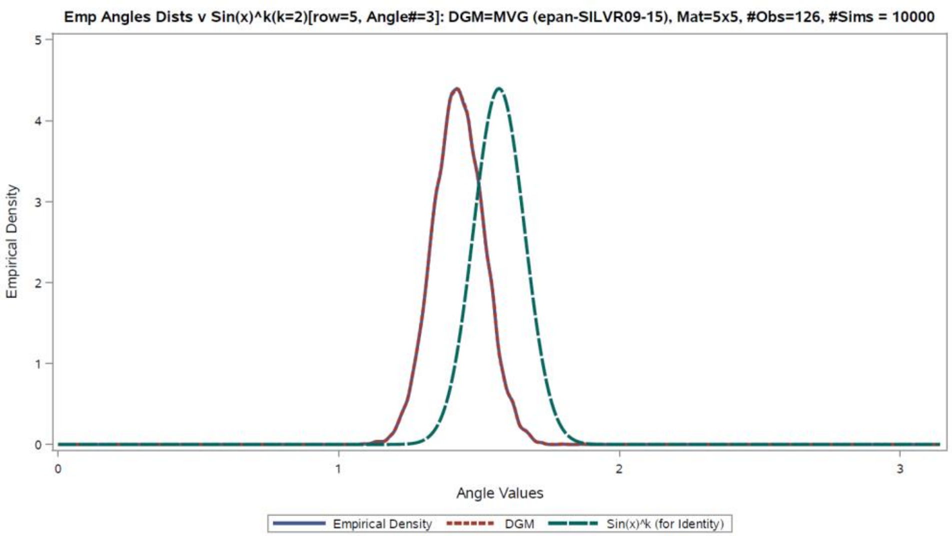


Angle Distributions

n = 6 **n = 126**



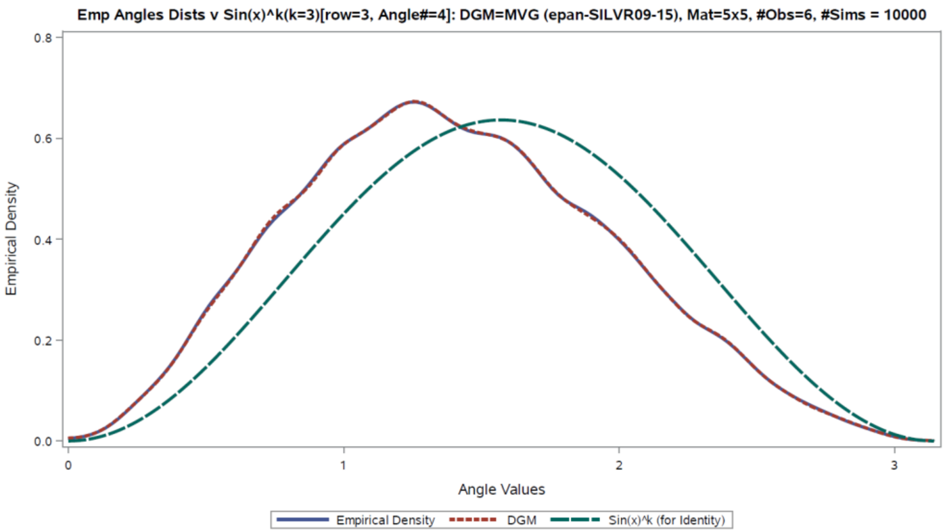
PRNG Seed = 12345
RunID = Bns5x5MVG



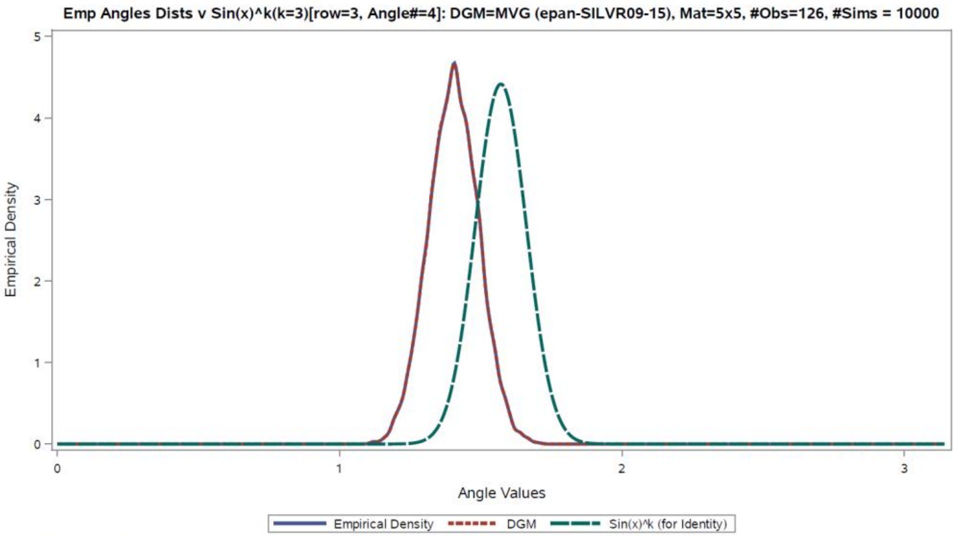
PRNG Seed = 12345
RunID = Bnb5x5MVG

Angle Distributions

n = 6 **n = 126**



PRNG Seed = 12345
RunID = Bns5x5MVG

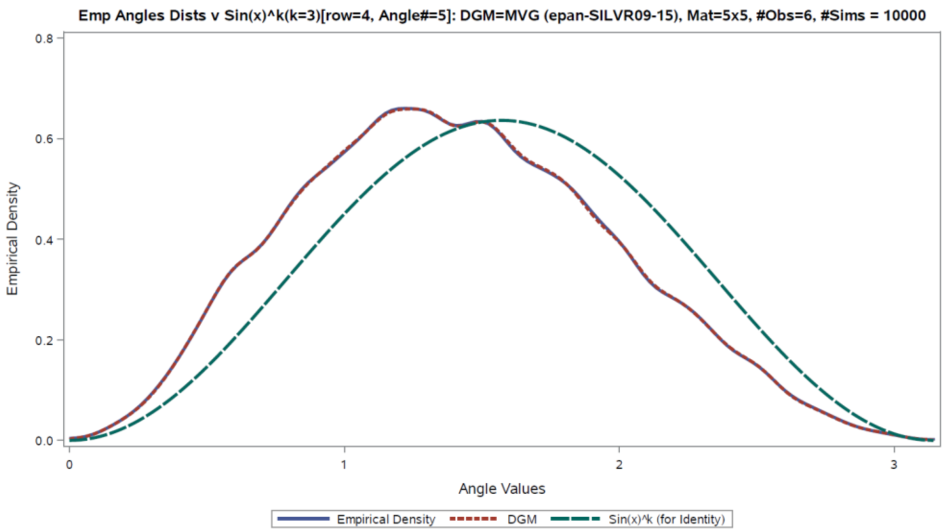


PRNG Seed = 12345
RunID = Bnb5x5MVG

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

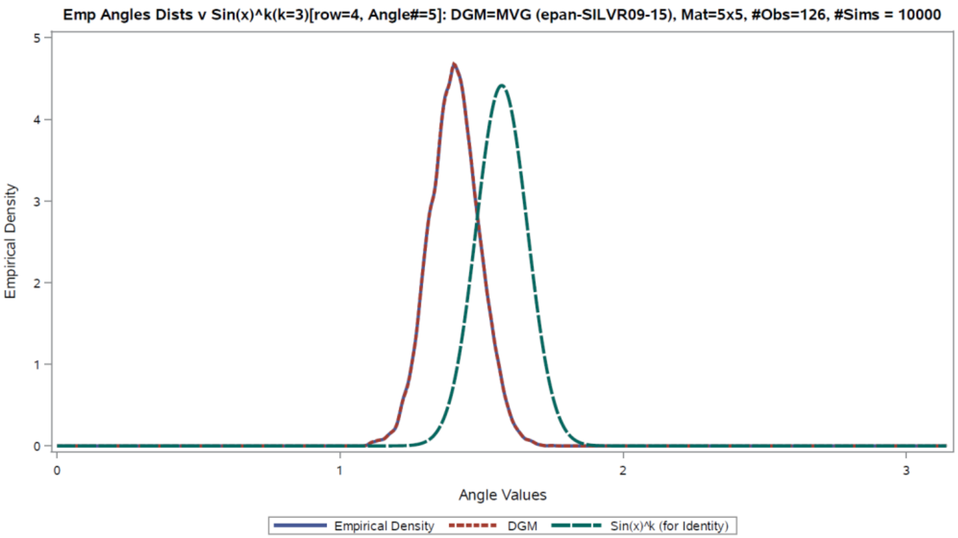
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Bns5x5MVG

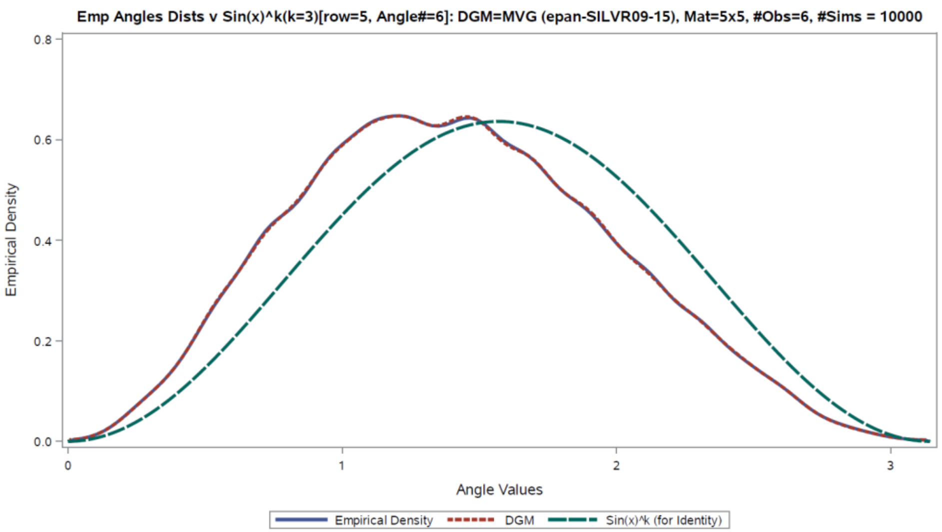
n = 126



PRNG Seed = 12345
RunID = Bnb5x5MVG

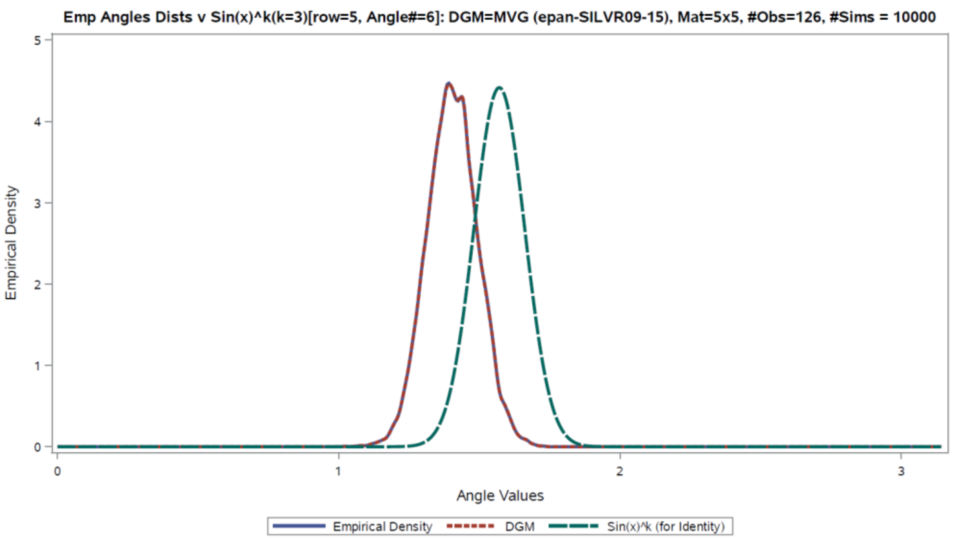
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Bns5x5MVG

n = 126

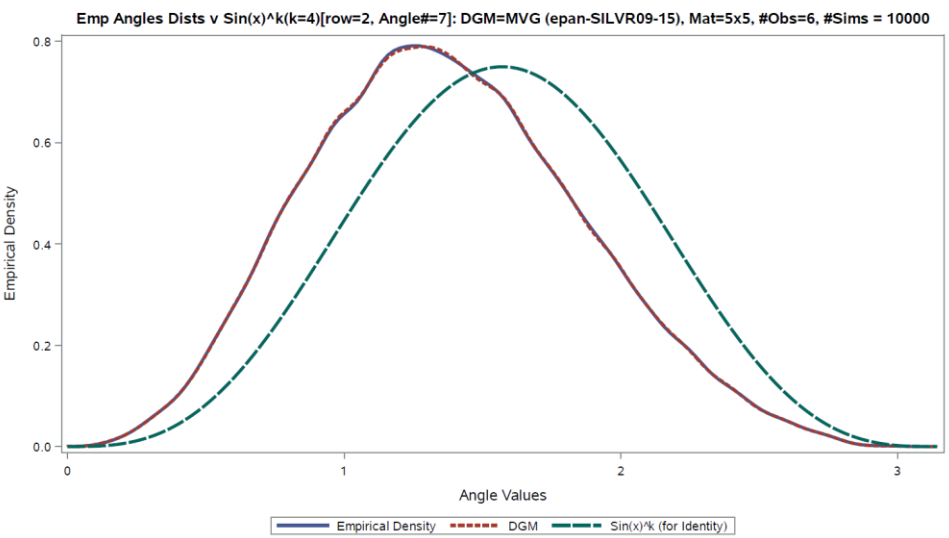


PRNG Seed = 12345
RunID = Bnb5x5MVG

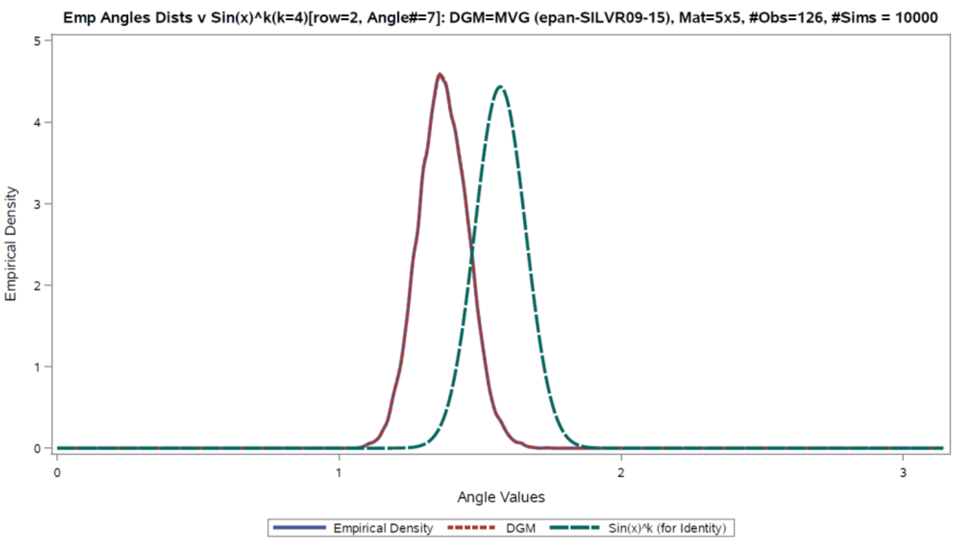
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

Angle Distributions

n = 6 | **n = 126**



PRNG Seed = 12345
RunID = Bns5x5MVG

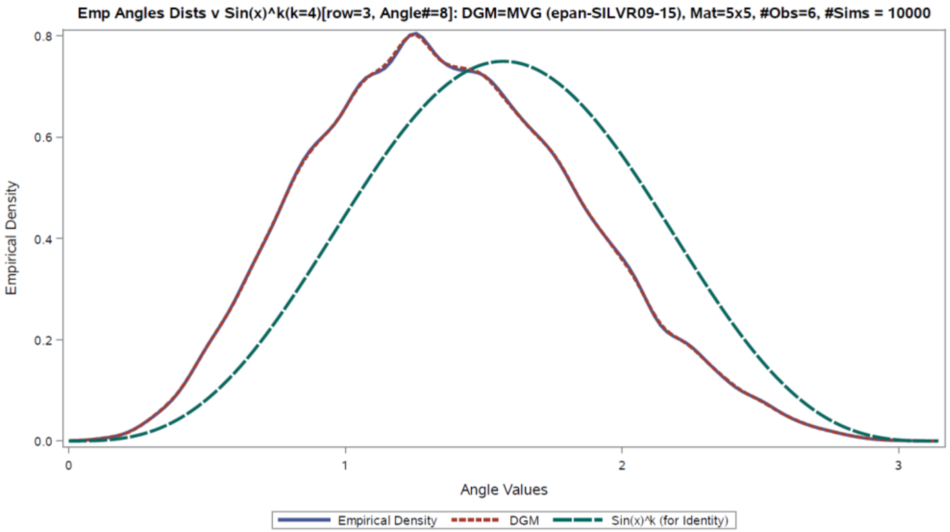


PRNG Seed = 12345
RunID = Bnb5x5MVG

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

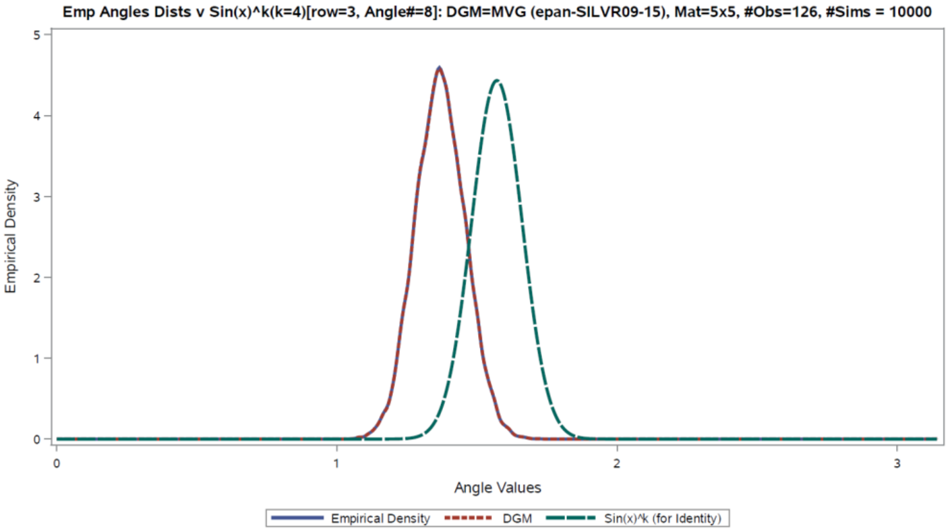
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Bns5x5MVG

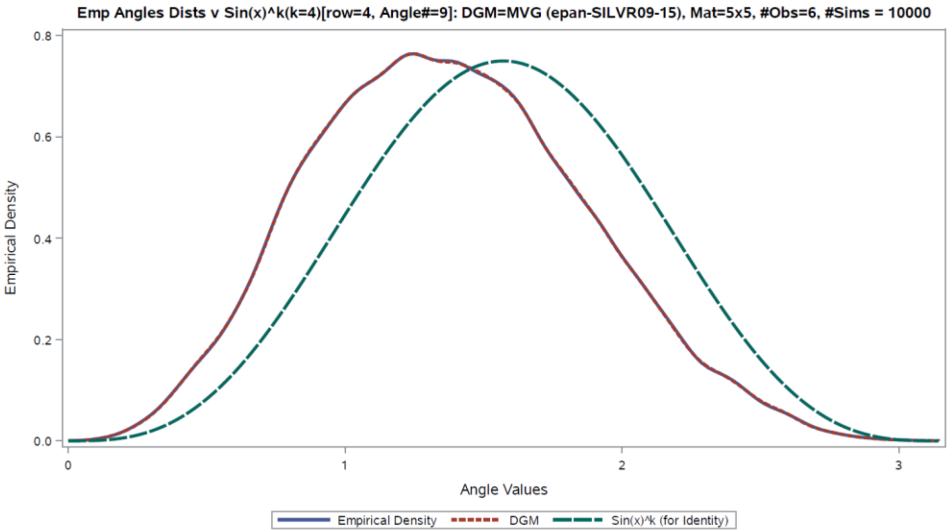
n = 126



PRNG Seed = 12345
RunID = Bnb5x5MVG

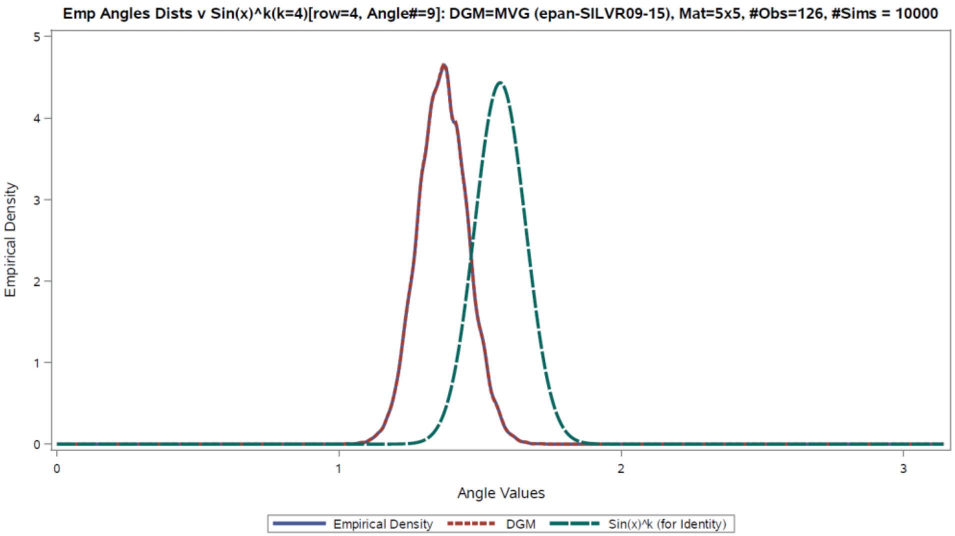
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Bns5x5MVG

n = 126

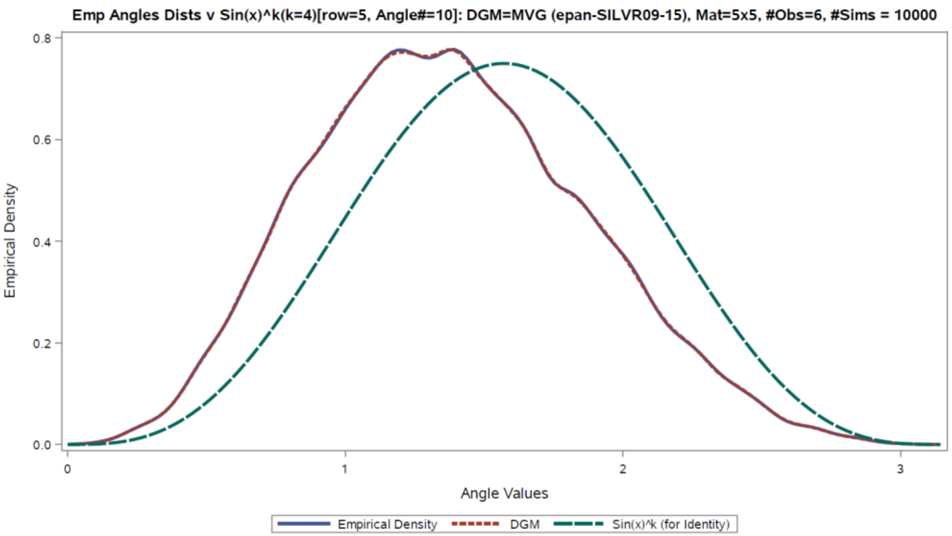


PRNG Seed = 12345
RunID = Bnb5x5MVG

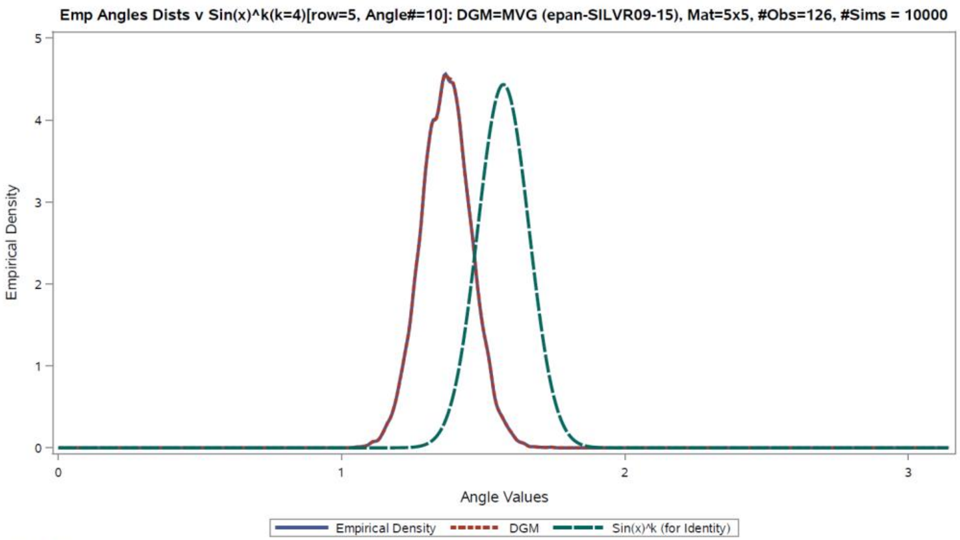
Angle Distributions

n = 6

n = 126



PRNG Seed = 12345
RunID = Bns5x5MVG



PRNG Seed = 12345
RunID = Bnb5x5MVG

Correlation Matrices to CDF Matrices

Correlation Matrices | **n=6** | **n=126**

1	1	1	1	1												
0	0.3	0.4	0.1	0.1		0.67	0.44	0.35	0.60	0.60		0.99	0.12	0.01	0.88	0.88
0	0.3	0.4	0.1	0.1		0.67	0.44	0.35	0.59	0.59		0.99	0.12	0.01	0.87	0.87
0	0.3	0.4	0.1	0.1		0.66	0.44	0.35	0.59	0.59		0.99	0.12	0.01	0.88	0.88
0	0.3	0.4	0.1	0.1		0.67	0.44	0.35	0.60	0.60		0.99	0.12	0.01	0.88	0.88
1	1	1	1	1												
0	0.3	0.4	0.1	0.4		0.63	0.48	0.44	0.57	0.36		0.97	0.23	0.08	0.81	0.00
0	0.3	0.4	0.1	0.4		0.62	0.48	0.44	0.56	0.36		0.97	0.23	0.08	0.81	0.00
0	0.3	0.4	0.1	0.4		0.62	0.47	0.43	0.56	0.36		0.97	0.23	0.08	0.80	0.00
1	1	1	1	1												
0	0.3	0.4	0.1	0.1		0.59	0.50	0.48	0.55	0.63		0.94	0.31	0.18	0.75	0.99
0	0.3	0.4	0.1	0.1		0.58	0.49	0.48	0.55	0.62		0.95	0.31	0.19	0.75	0.99
1	1	1	1	1												
0	0.3	0.4	0.1	0.3		0.57	0.52	0.51	0.54	0.52		0.91	0.35	0.26	0.69	0.35
1	1	1	1	1												

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

CDF Matrices to Correlation Matrices

CDF Matrices | **n=6** | **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.23	0.35	0.10	0.41	-0.44		0.20	0.22	0.18	0.24	0.09
0.5	0.4	0.6	0.35	0.9		0.22	0.35	0.09	0.41	-0.44		0.20	0.22	0.18	0.23	0.09
0.5	0.4	0.6	0.35	0.9		0.22	0.34	0.08	0.40	-0.44		0.20	0.22	0.18	0.24	0.09
0.5	0.4	0.6	0.35	0.9		0.22	0.35	0.09	0.41	-0.45		0.20	0.22	0.18	0.23	0.09
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.24	0.42	0.05	0.20	-0.07		0.20	0.23	0.17	0.19	0.10
0.5	0.4	0.6	0.6	0.8		0.23	0.42	0.04	0.19	-0.07		0.20	0.23	0.17	0.19	0.10
0.5	0.4	0.6	0.6	0.8		0.22	0.41	0.04	0.19	-0.07		0.20	0.23	0.17	0.19	0.10
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.25	0.51	-0.01	0.48	0.67		0.20	0.23	0.17	0.23	0.20
0.5	0.4	0.6	0.4	0.3		0.24	0.51	-0.02	0.47	0.67		0.20	0.24	0.17	0.23	0.20
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.30	0.65	-0.08	0.32	0.01		0.20	0.24	0.16	0.20	0.03
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case B

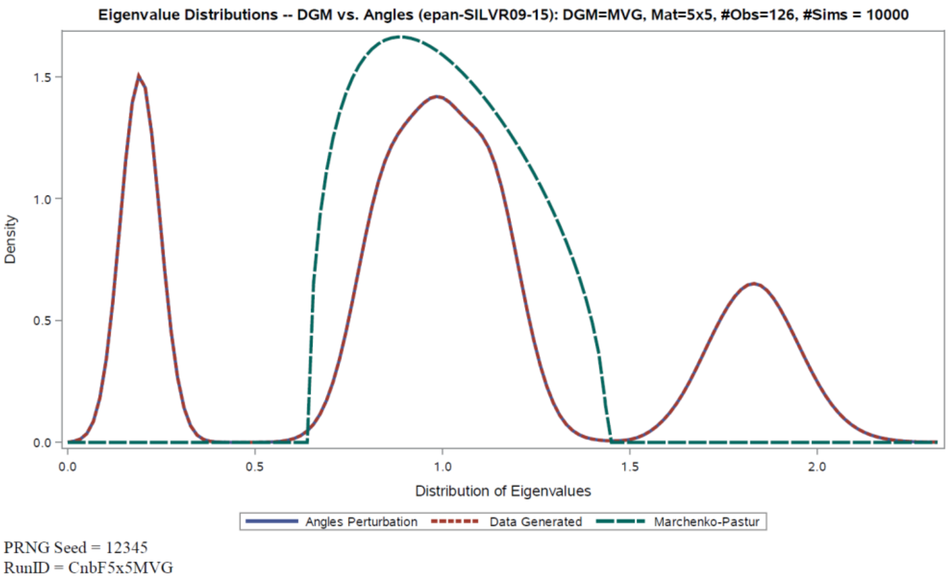
CDF %Shift Matrices to Correlation Matrices

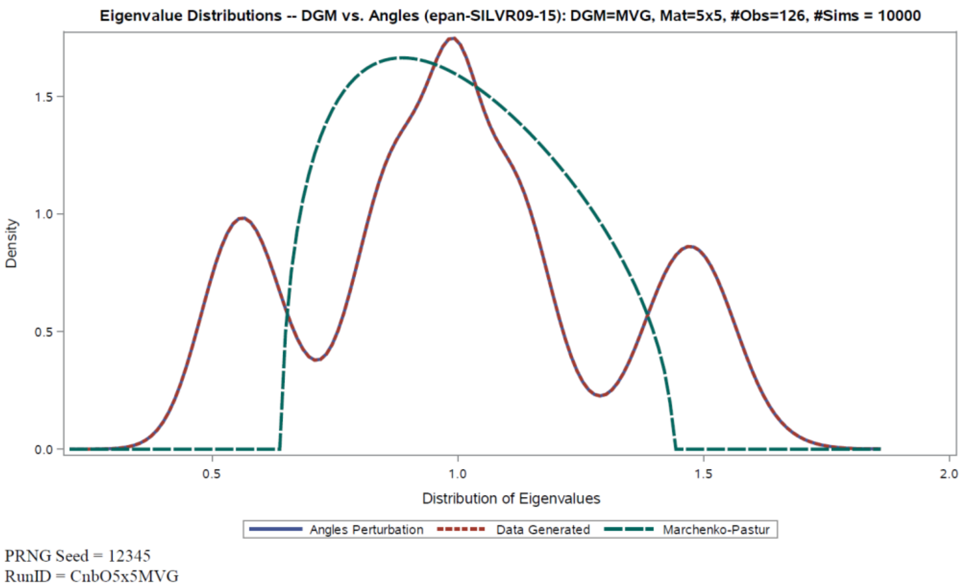
CDF %Shift Matrices **n=6** **n=126**

					lnMatPr	-2.373	-2.092	-7.511	-6.385	-14.597		-2.261	-2.202	-7.050	-6.814	-13.154
					FNorm	1.011	0.876	2.199	1.748	2.684		0.128	0.125	0.347	0.334	0.548
					Rnk_lnMat	2	1	4	3	5		2	1	4	3	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.31	0.06	0.51	-0.16	0.58		0.22	0.18	0.26	0.14	0.27
20	-20	50	-50	60		0.31	0.06	0.51	-0.16	0.58		0.22	0.18	0.26	0.14	0.27
20	-20	50	-50	60		0.31	0.05	0.51	-0.16	0.58		0.22	0.18	0.26	0.14	0.27
20	-20	50	-50	60		0.31	0.05	0.51	-0.17	0.57		0.22	0.18	0.26	0.14	0.27
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.38	0.01	0.66	-0.22	0.80		0.23	0.17	0.28	0.13	0.31
20	-20	50	-50	70		0.38	0.01	0.66	-0.22	0.79		0.23	0.17	0.28	0.13	0.31
20	-20	50	-50	70		0.38	0.00	0.65	-0.23	0.79		0.23	0.17	0.28	0.12	0.31
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.47	-0.05	0.80	-0.25	0.95		0.23	0.17	0.29	0.11	0.35
20	-20	50	-50	80		0.46	-0.06	0.80	-0.25	0.95		0.23	0.17	0.29	0.11	0.36
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.59	-0.16	0.93	-0.26	1.00		0.24	0.16	0.31	0.10	0.41
						1	1	1	1	1		1	1	1	1	1

Spectral Distributions

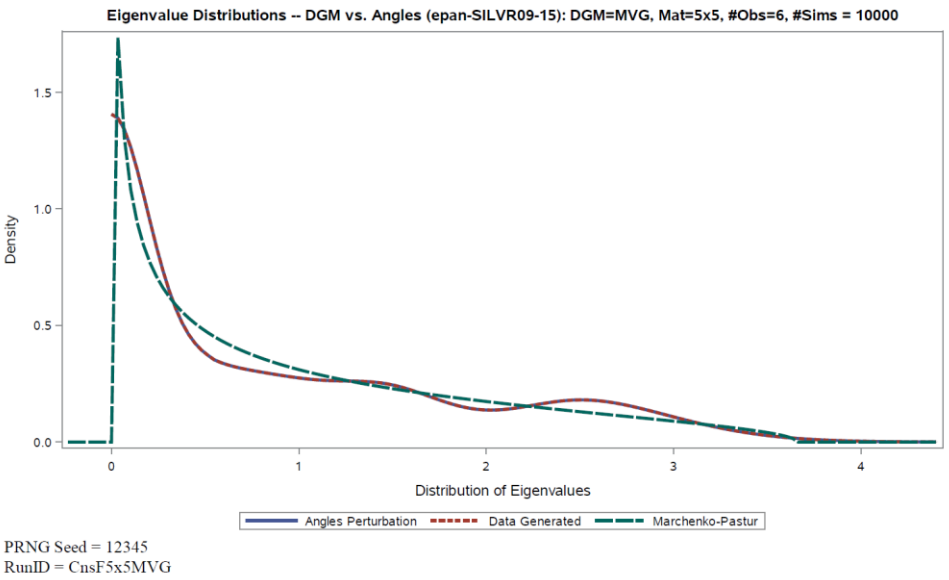
one factor = 0.4 **(n = 126)** **one cell = 0.4**

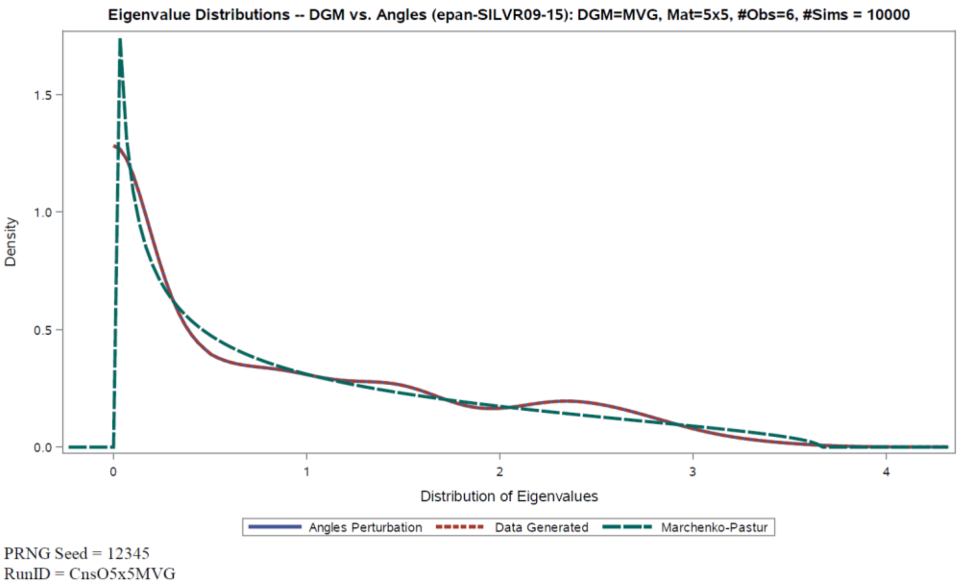




Spectral Distributions

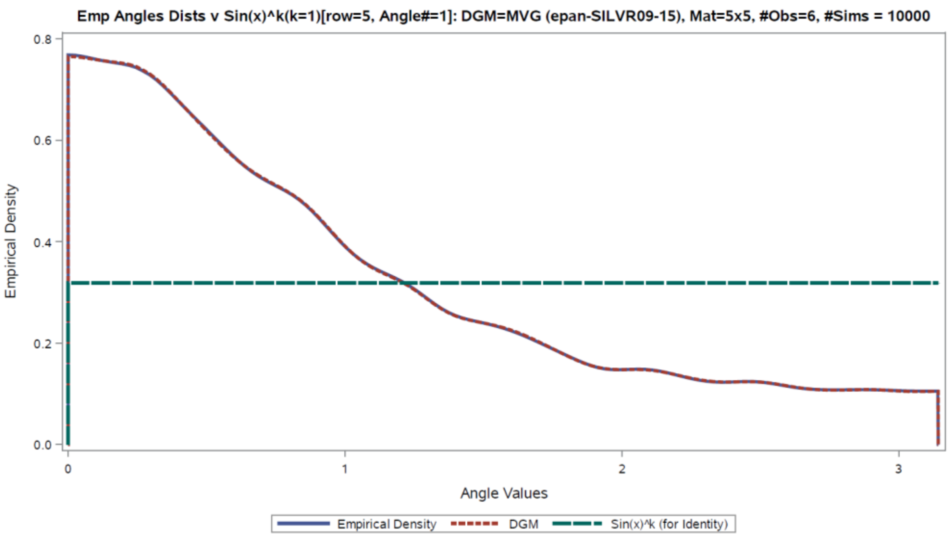
one factor = 0.4 **(n = 6)** **one cell = 0.4**



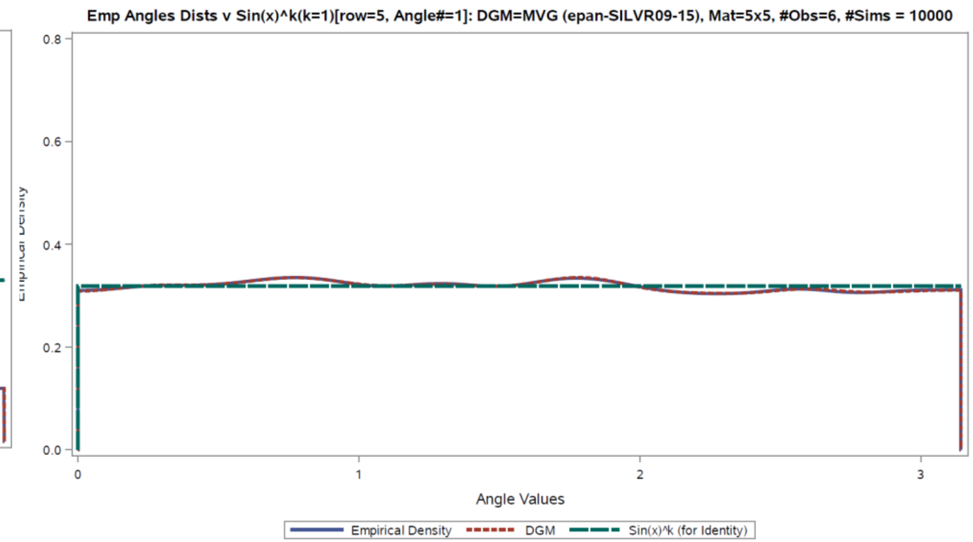


Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**



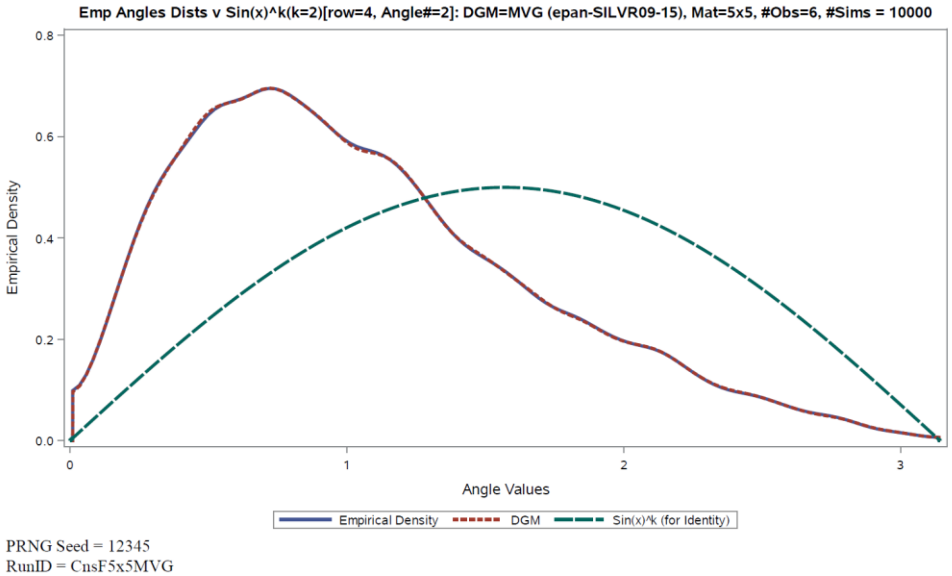
PRNG Seed = 12345
RunID = CnsF5x5MVG

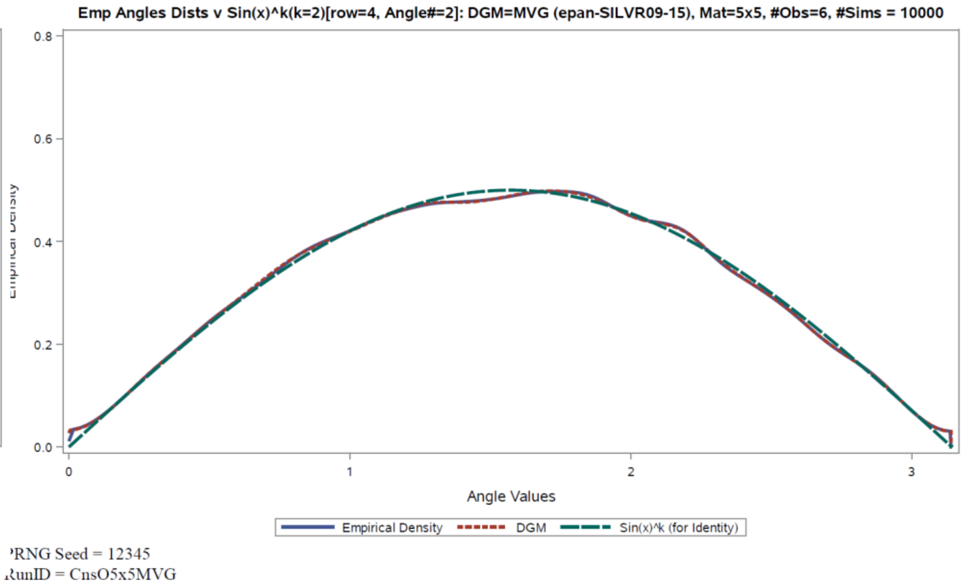


'RNG Seed = 12345
RunID = CnsO5x5MVG

Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**

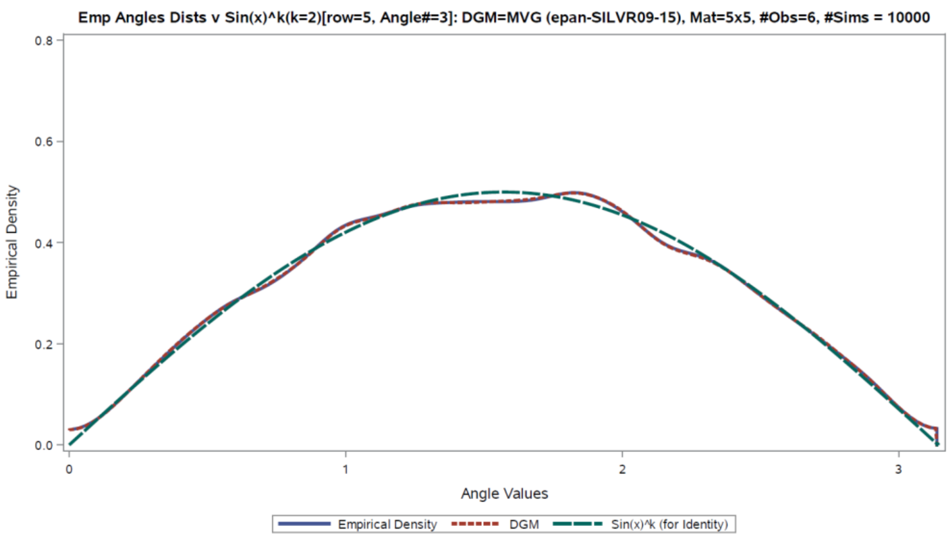




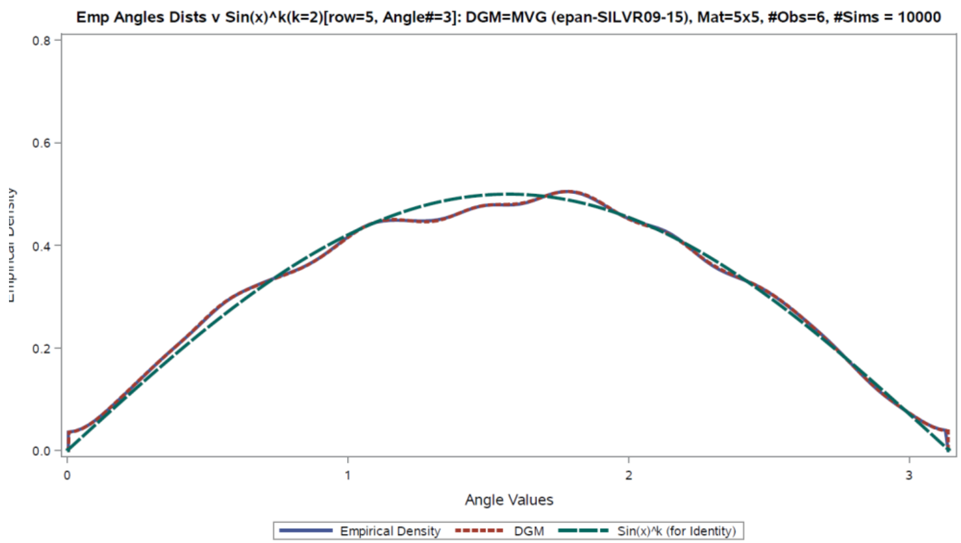
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**



PRNG Seed = 12345
RunID = CnsF5x5MVG

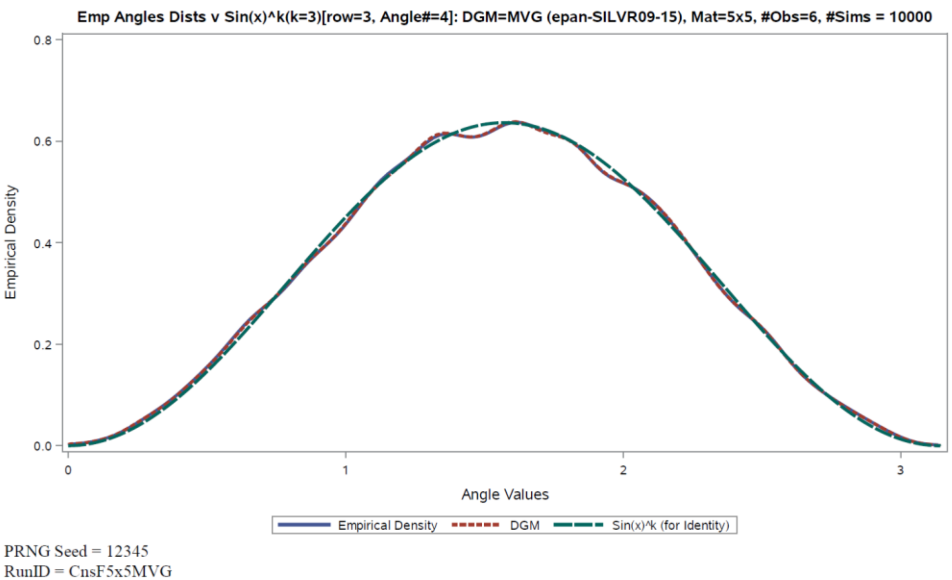


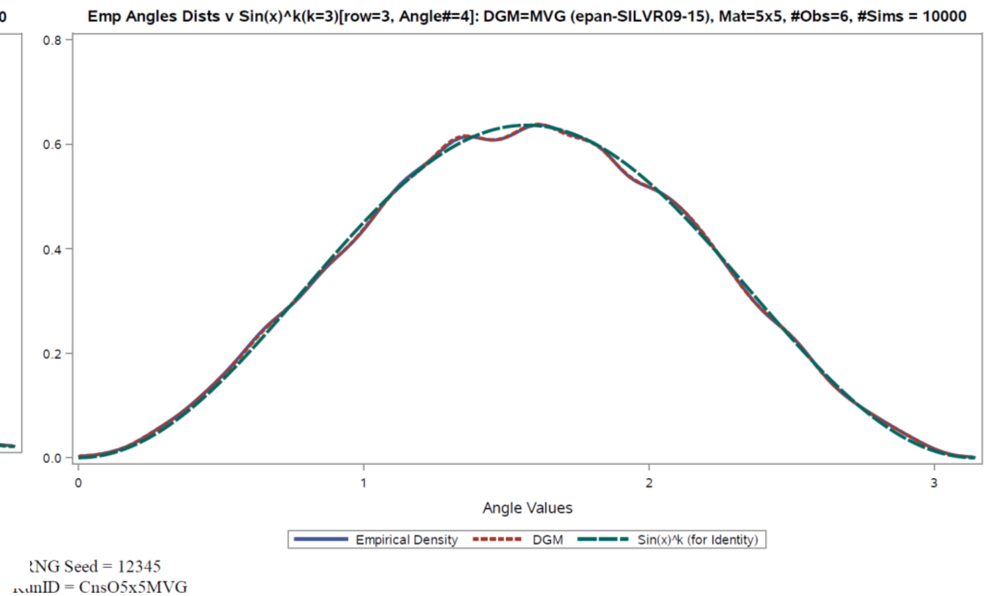
PRNG Seed = 12345
RunID = CnsO5x5MVG

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**

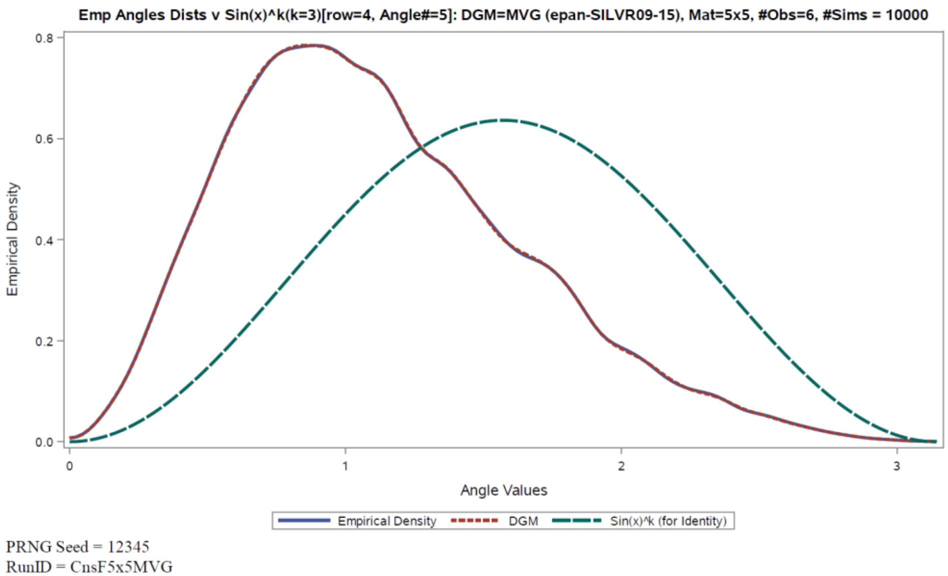


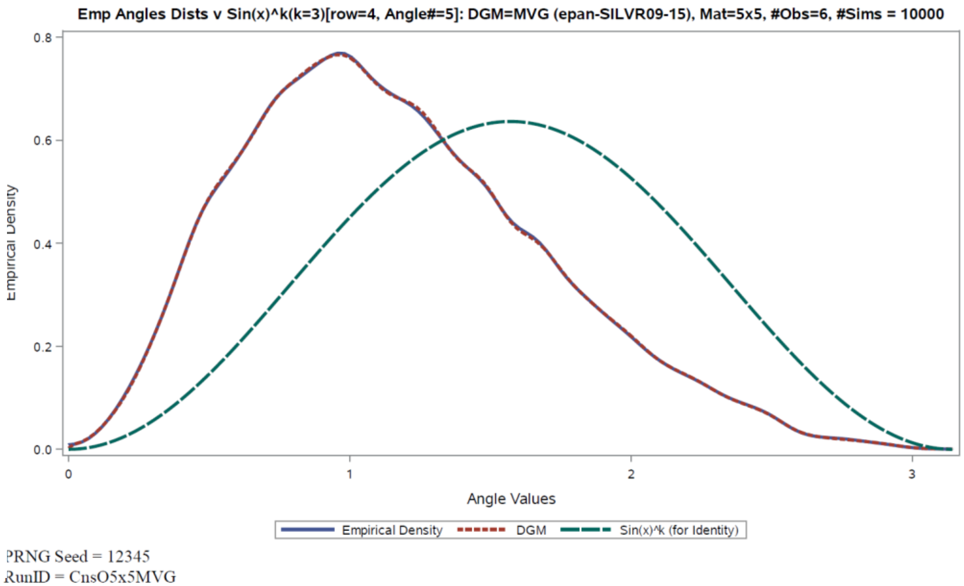


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**

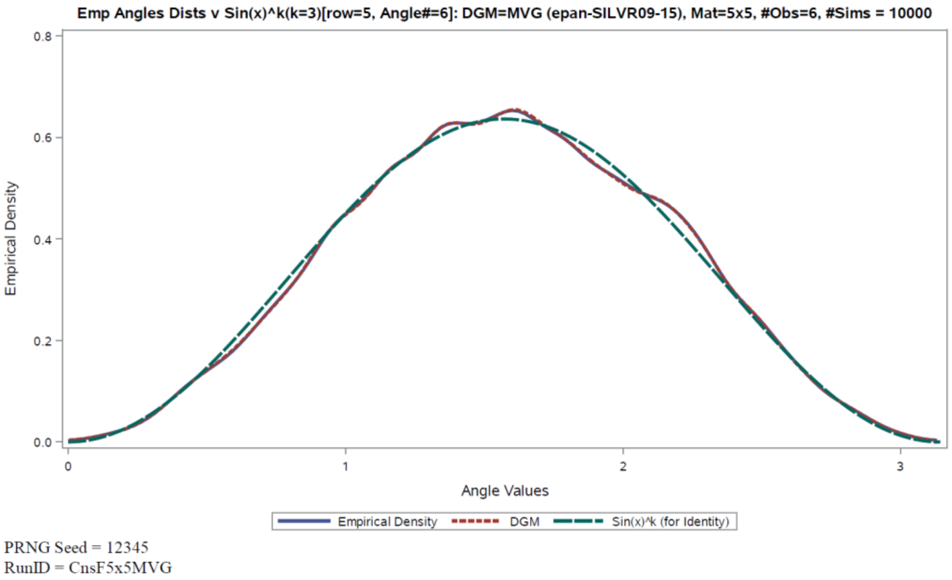


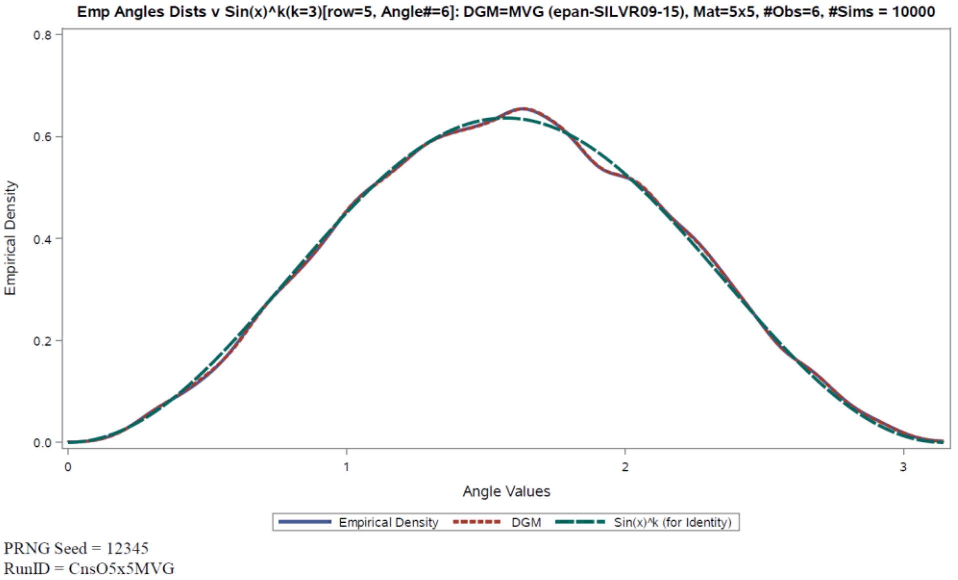


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**

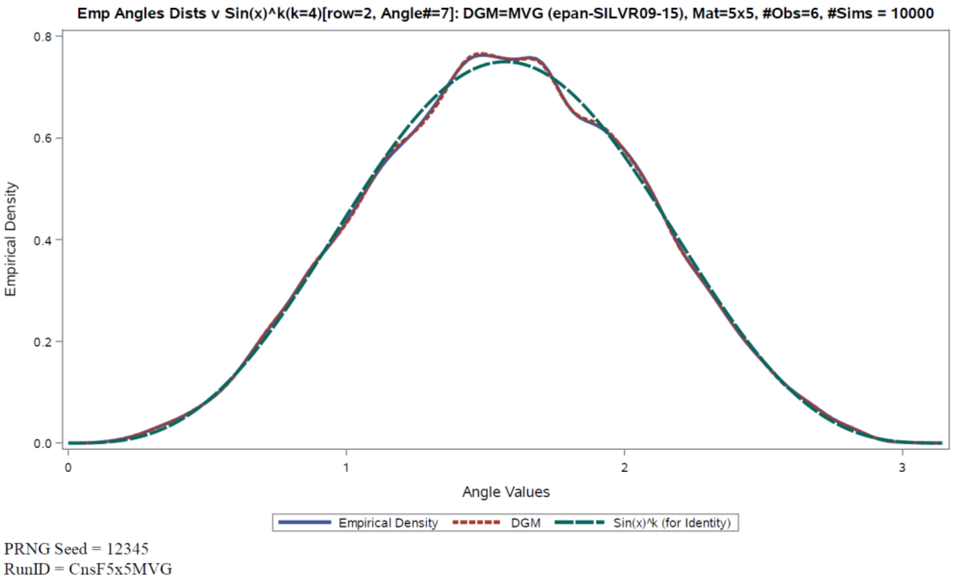


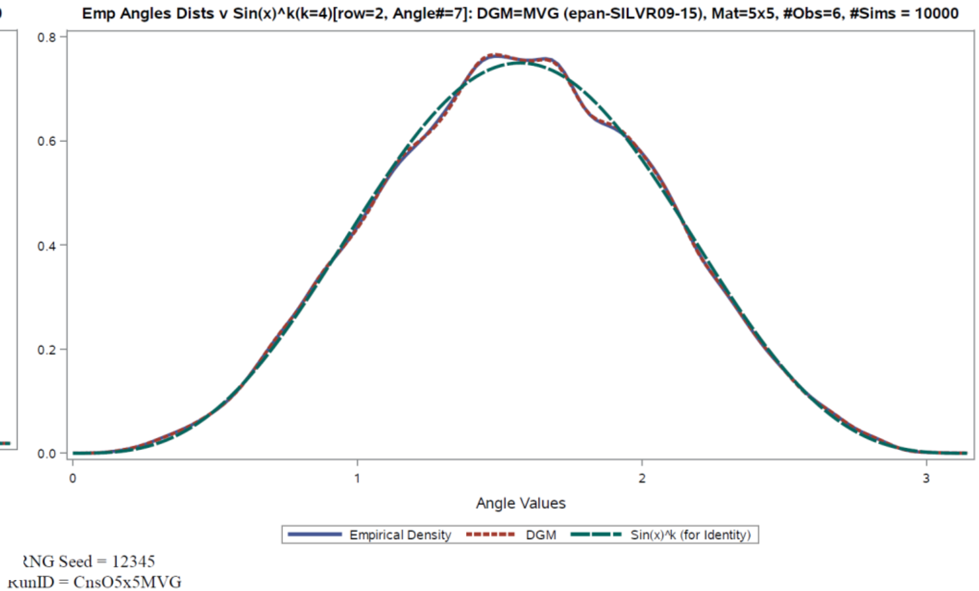


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Angle Distributions

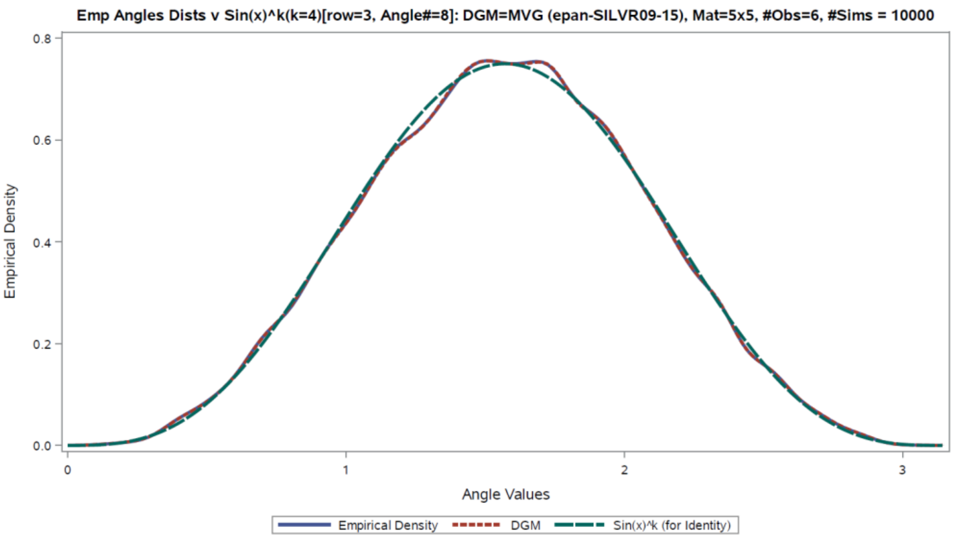
one factor = 0.4 **(n = 6)** **one cell = 0.4**



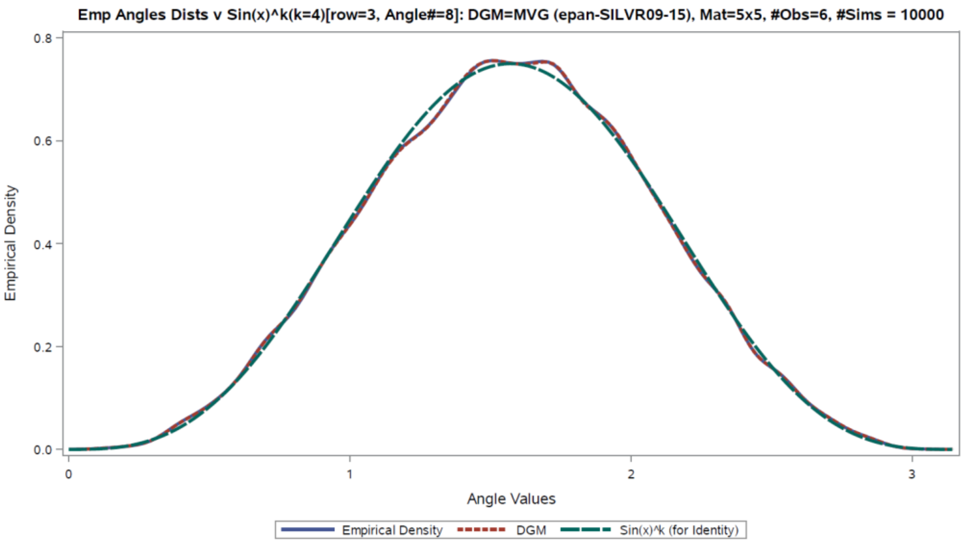


Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**



PRNG Seed = 12345
RunID = CnsF5x5MVG

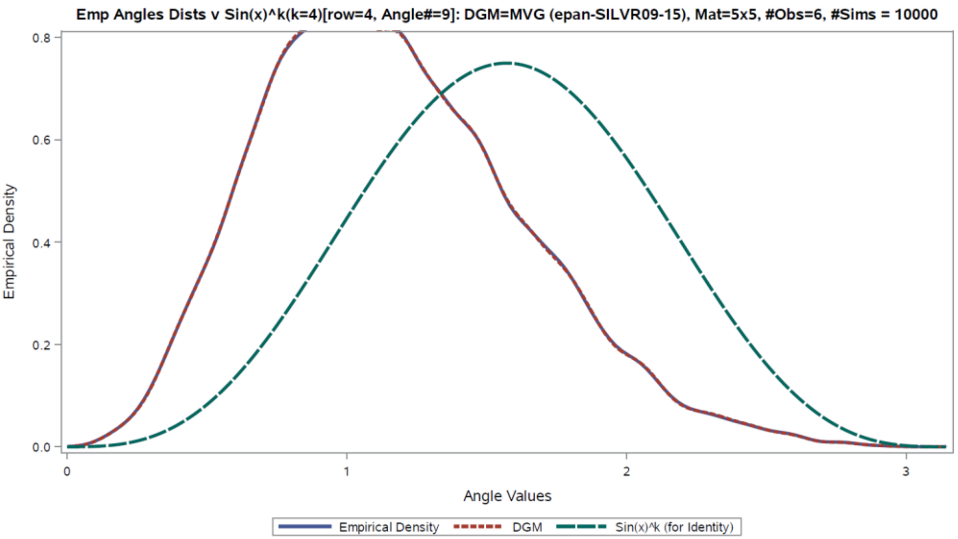


PRNG Seed = 12345
RunID = CnsO5x5MVG

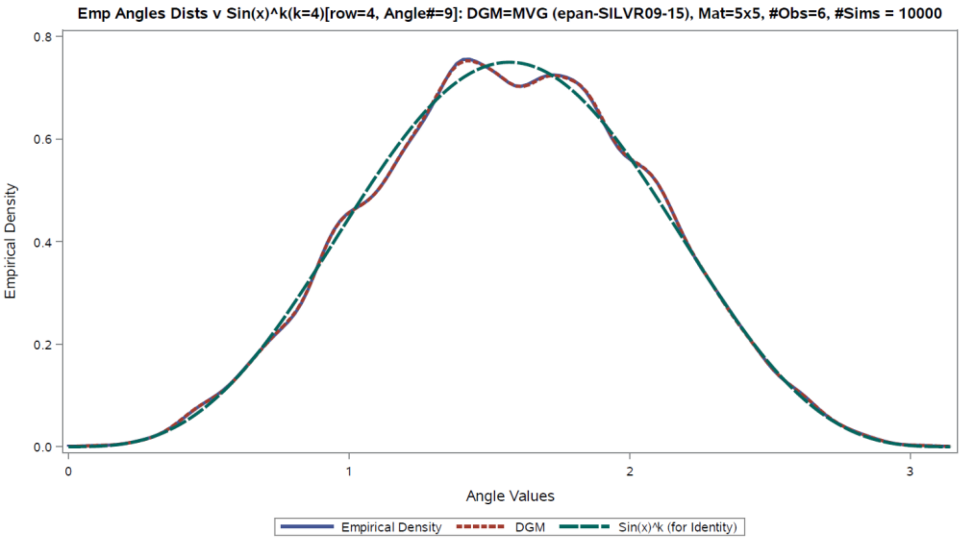
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Angle Distributions

one factor = 0.4 (n = 6) one cell = 0.4



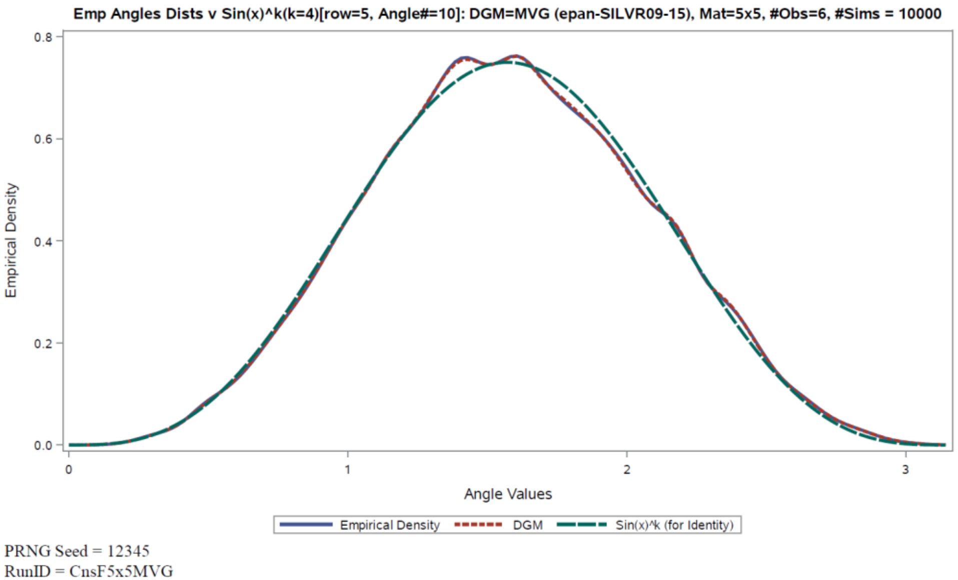
PRNG Seed = 12345
RunID = CnsF5x5MVG

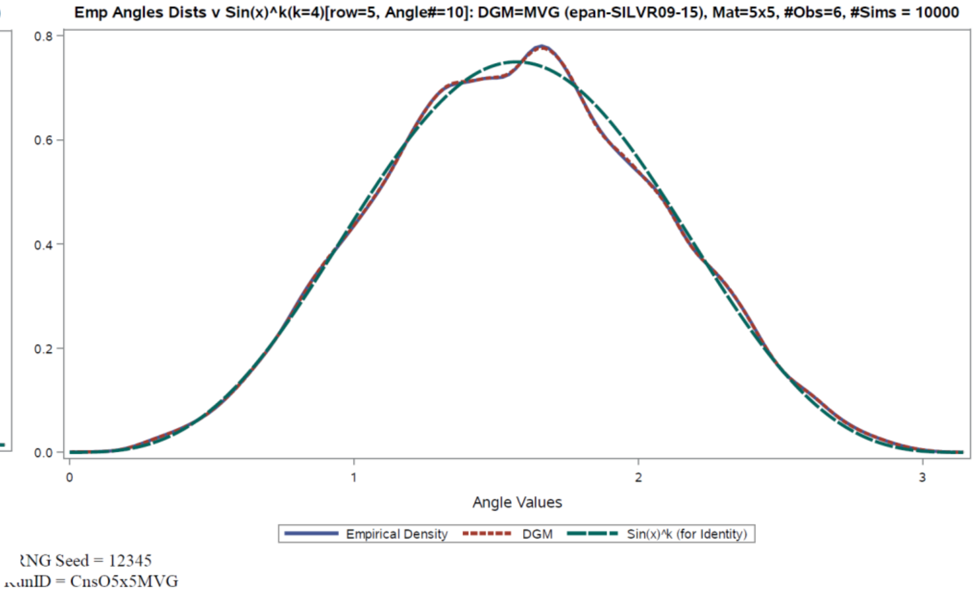


PRNG Seed = 12345
RunID = CnsO5x5MVG

Angle Distributions

one factor = 0.4 **(n = 6)** **one cell = 0.4**





XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

Correlation Matrices to CDF Matrices

Correlation Matrices | **one factor = 0.4** | **(n = 6)** | **one cell = 0.4**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.43	0.35	0.58	0.58		0.50	0.43	0.35	0.58	0.58
0	0.1	0.2	-0.1	-0.1		0.50	0.42	0.35	0.57	0.57		0.50	0.42	0.35	0.57	0.57
0	0.1	0.2	-0.1	-0.1		0.81	0.76	0.69	0.86	0.86		0.50	0.42	0.35	0.57	0.57
0	0.1	0.2	-0.1	-0.1		0.50	0.43	0.35	0.58	0.58		0.50	0.42	0.35	0.58	0.58
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.50	0.44	0.40	0.57	0.38		0.50	0.44	0.40	0.57	0.38
0	0.1	0.2	-0.1	0.2		0.80	0.76	0.73	0.84	0.71		0.78	0.74	0.70	0.83	0.68
0	0.1	0.2	-0.1	0.2		0.49	0.44	0.39	0.57	0.37		0.49	0.44	0.39	0.57	0.37
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.80	0.77	0.74	0.84	0.84		0.50	0.46	0.43	0.56	0.58
0	0.1	0.2	-0.1	-0.1		0.50	0.46	0.43	0.56	0.57		0.50	0.46	0.43	0.56	0.57
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.79	0.77	0.75	0.82	0.73		0.50	0.48	0.47	0.55	0.43
1	1	1	1	1												

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

CDF Matrices to Correlation Matrices

CDF Matrices **one factor = 0.4** **(n = 6)** **one cell = 0.4**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.00	0.13	-0.13	0.20	-0.61		0.00	0.13	-0.13	0.20	-0.61
0.5	0.4	0.6	0.35	0.9		0.00	0.13	-0.14	0.19	-0.60		0.00	0.13	-0.14	0.19	-0.60
0.5	0.4	0.6	0.35	0.9		0.44	0.54	0.32	0.59	-0.22		0.00	0.13	-0.14	0.20	-0.61
0.5	0.4	0.6	0.35	0.9		0.00	0.14	-0.13	0.20	-0.62		0.00	0.14	-0.13	0.21	-0.62
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.00	0.18	-0.14	-0.11	0.05		0.00	0.18	-0.14	-0.11	0.05
0.5	0.4	0.6	0.6	0.8		0.45	0.58	0.31	0.41	0.14		0.46	0.58	0.33	0.35	0.34
0.5	0.4	0.6	0.6	0.8		-0.01	0.16	-0.14	-0.12	0.06		-0.01	0.16	-0.14	-0.12	0.07
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.45	0.61	0.27	0.57	0.67		0.00	0.28	-0.21	0.18	0.61
0.5	0.4	0.6	0.4	0.3		0.00	0.24	-0.15	0.26	0.72		0.00	0.23	-0.16	0.25	0.72
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.46	0.63	0.27	0.51	0.01		0.01	0.38	-0.24	-0.09	0.04
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case C

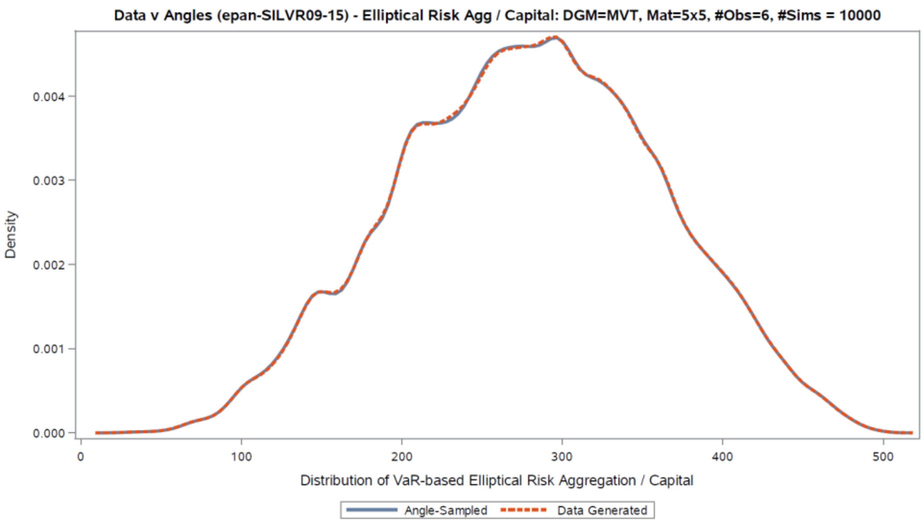
CDF %Shift Matrices to Correlation Matrices

CDF %Shift Matrices | **one factor = 0.4** | **(n = 6)** | **one cell = 0.4**

					lnMatPr	-2.434	-2.040	-7.821	-6.220	-14.443		-2.259	-2.205	-7.051	-6.830	-13.156
					FNorm	0.806	0.698	1.905	1.512	2.478		0.936	0.698	2.273	1.229	2.936
					Rnk_lnMat	2	1	4	3	5		2	1	4	3	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.13	-0.13	0.35	-0.35	0.43		0.13	-0.13	0.35	-0.35	0.43
20	-20	50	-50	60		0.13	-0.14	0.35	-0.34	0.42		0.13	-0.14	0.35	-0.34	0.42
20	-20	50	-50	60		0.49	0.25	0.65	0.05	0.70		0.13	-0.14	0.34	-0.35	0.42
20	-20	50	-50	60		0.13	-0.13	0.34	-0.35	0.42		0.14	-0.13	0.35	-0.35	0.42
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.18	-0.13	0.48	-0.24	0.67		0.18	-0.13	0.48	-0.24	0.67
20	-20	50	-50	70		0.53	0.23	0.73	0.01	0.82		0.52	0.26	0.73	0.12	0.84
20	-20	50	-50	70		0.17	-0.14	0.46	-0.24	0.65		0.17	-0.14	0.47	-0.24	0.65
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.58	0.19	0.80	-0.04	0.89		0.28	-0.20	0.67	-0.28	0.89
20	-20	50	-50	80		0.24	-0.15	0.64	-0.09	0.89		0.24	-0.15	0.64	-0.10	0.90
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.62	0.10	0.82	-0.16	0.89		0.38	-0.24	0.82	-0.10	0.96
						1	1	1	1	1		1	1	1	1	1

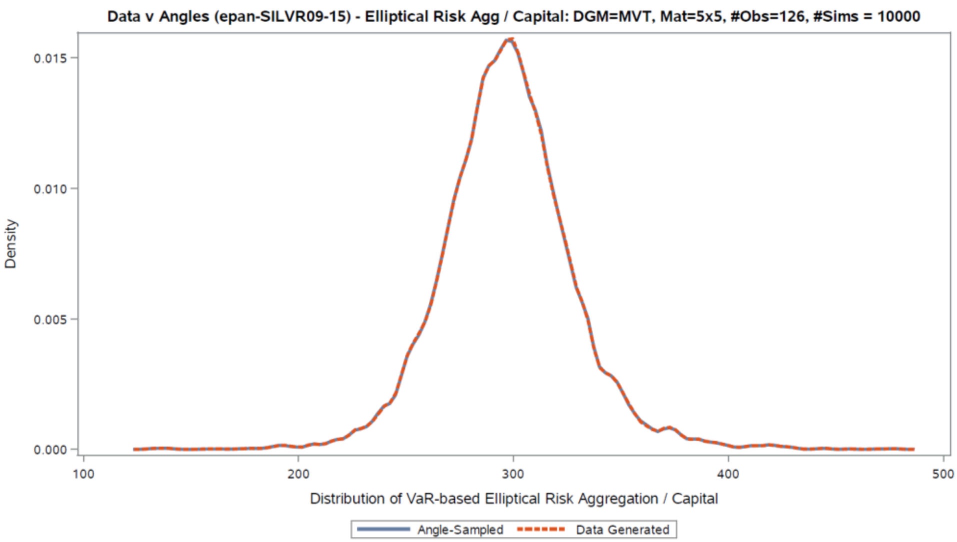
Elliptical Capital

n = 6



PRNG Seed = 12345
RunID = Dns5x5MVT

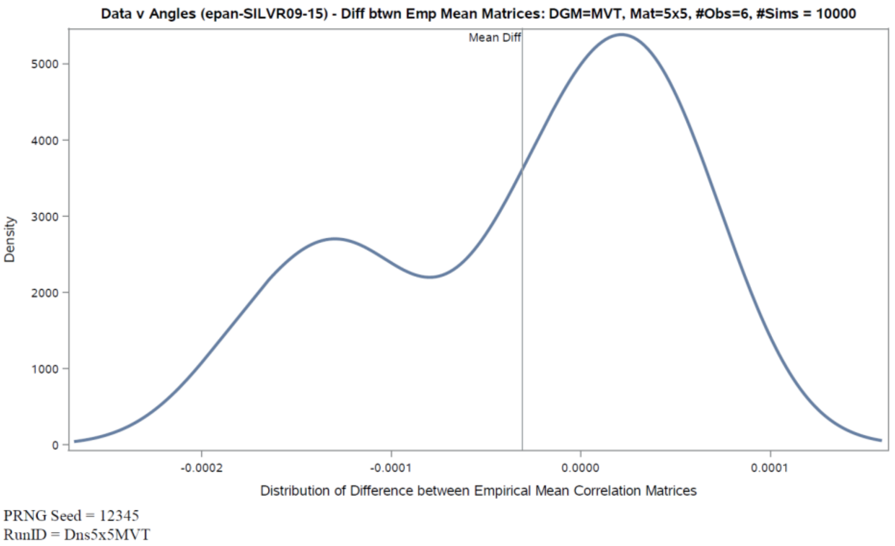
n = 126

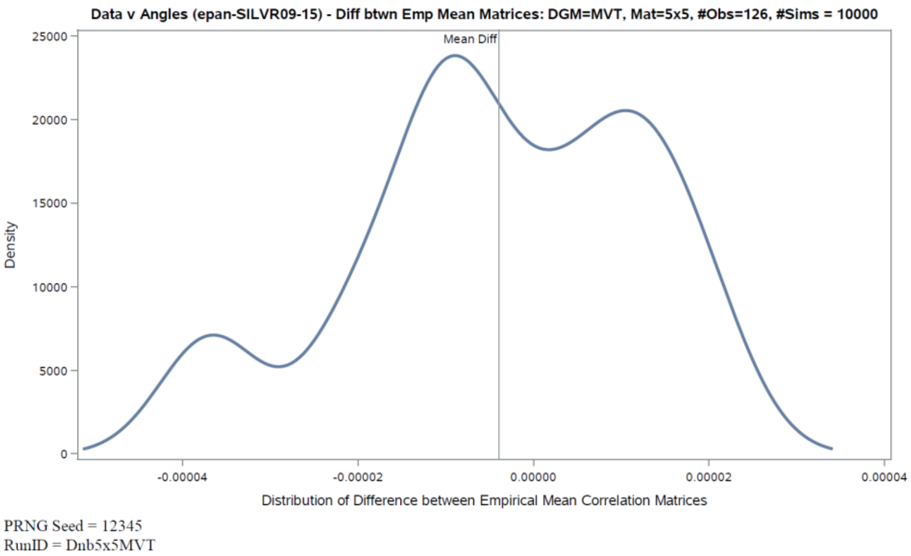


PRNG Seed = 12345
RunID = Dnb5x5MVT

Difference Between Mean Empirical Matrices

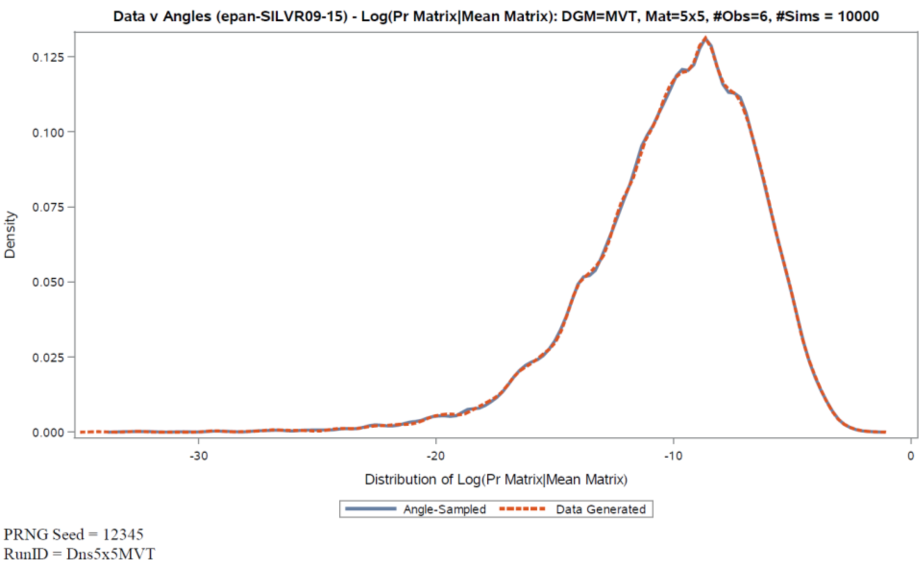
n = 6 **n = 126**





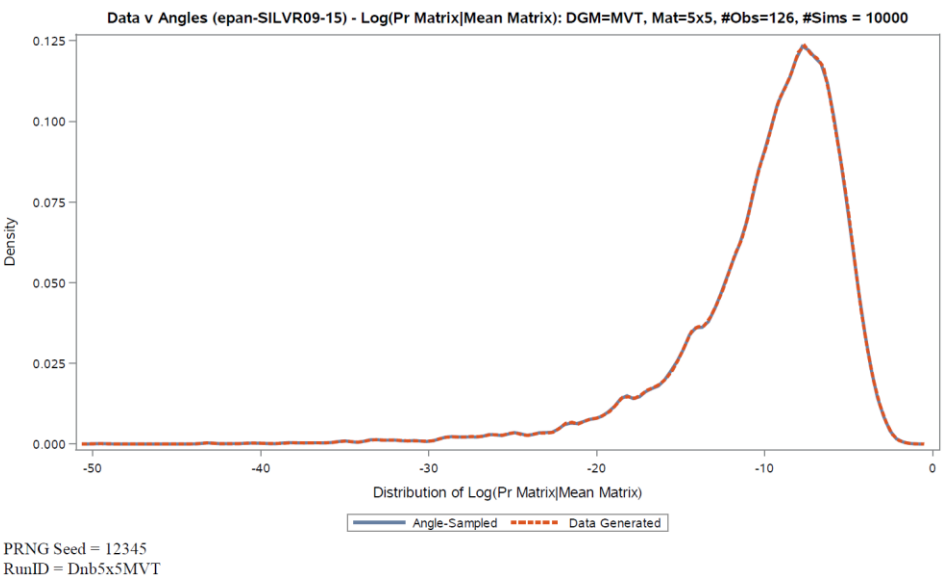
LNP

n = 6



Data v Angles (epan-SILVR09-15) - Log(Pr Matrix|Mean Matrix): DGM=MVT, Mat=5x5, #Obs=6, #Sims = 10000
Density
0.125
0.100
0.075
0.050
0.025
0.000
-30
-20
-10
0
Distribution of Log(Pr Matrix|Mean Matrix)
Angle-Sampled
Data Generated
PRNG Seed = 12345
RunID = Dns5x5MVT

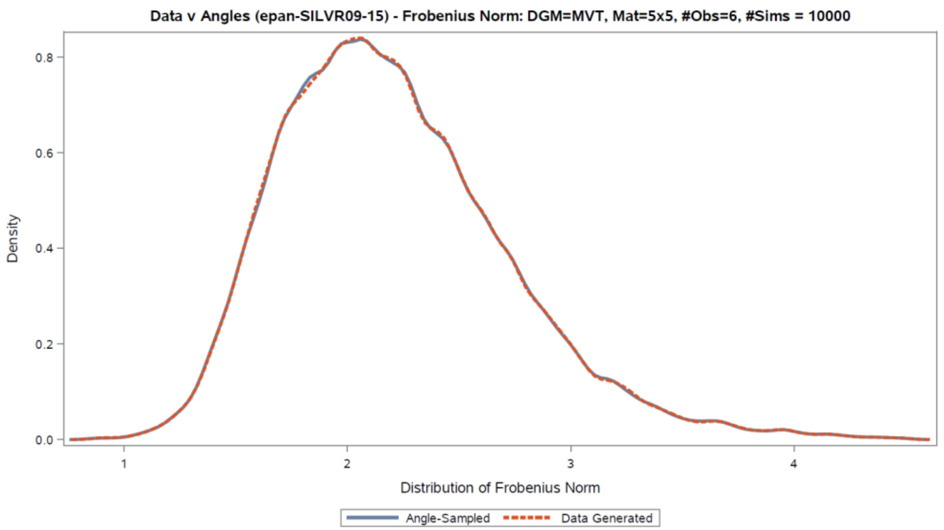
n = 126



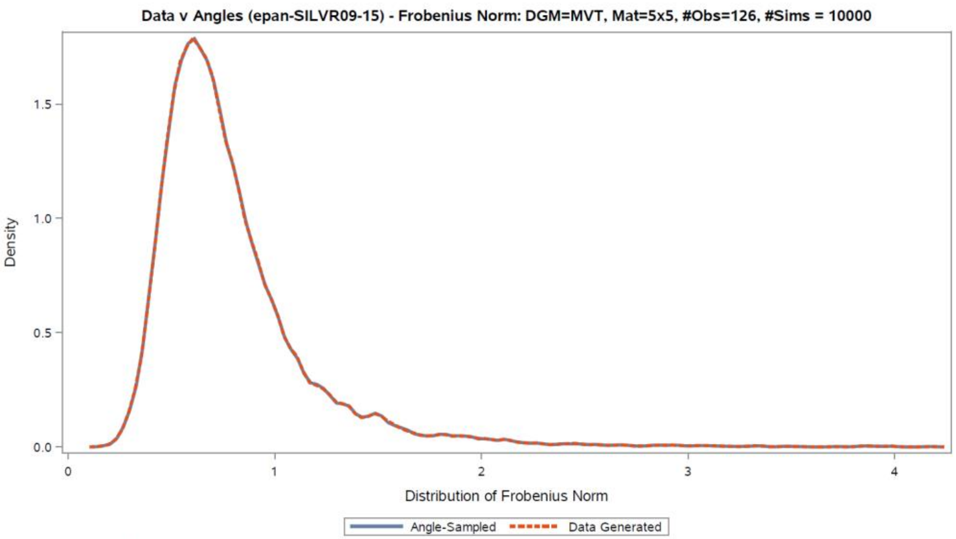
Data v Angles (epan-SILVR09-15) - Log(Pr Matrix|Mean Matrix): DGM=MVT, Mat=5x5, #Obs=126, #Sims = 10000
Density
0.125
0.100
0.075
0.050
0.025
0.000
-50
-40
-30
-20
-10
0
Distribution of Log(Pr Matrix|Mean Matrix)
Angle-Sampled
Data Generated
PRNG Seed = 12345
RunID = Dnb5x5MVT

Euclidian/Frobenius Norm

n = 6 | **n = 126**



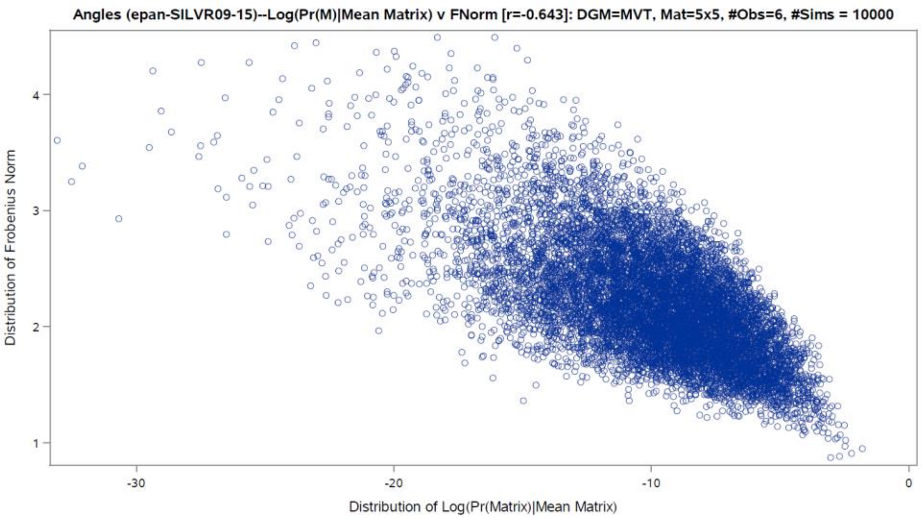
PRNG Seed = 12345
RunID = Dns5x5MVT



PRNG Seed = 12345
RunID = Dnb5x5MVT

LNP v Euclidian/Frobenius Norm

n = 6



PRNG Seed = 12345
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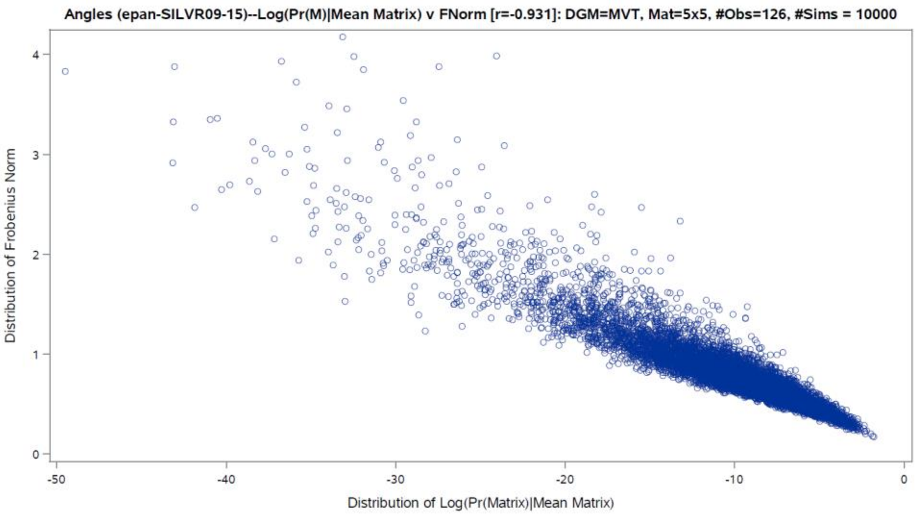
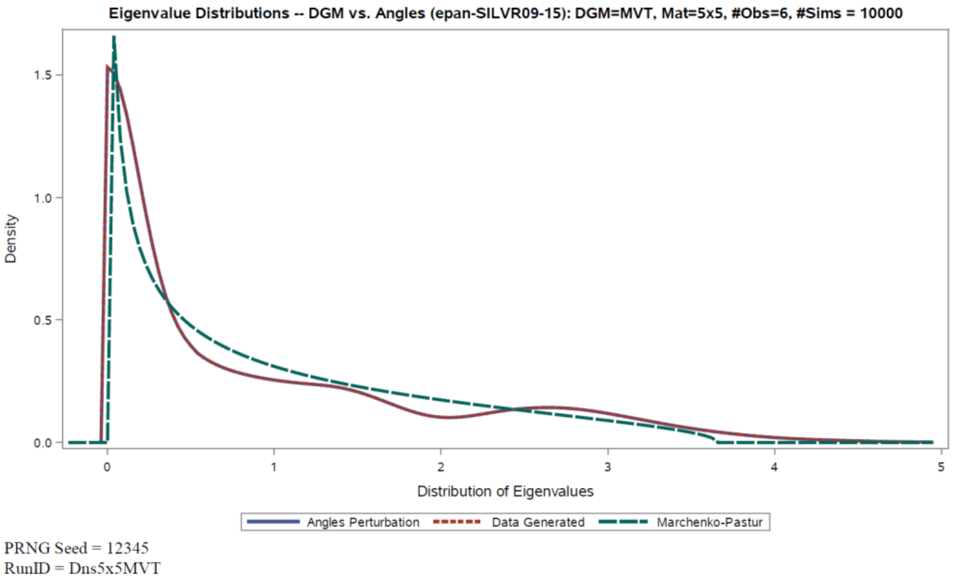
n = 126

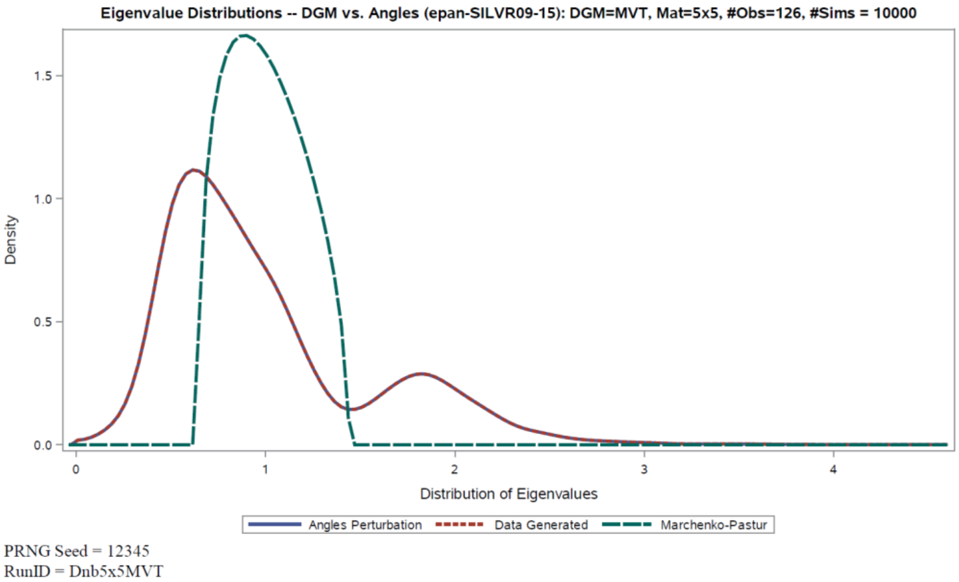


PRNG Seed = 12345
RunID = Dnb5x5MVT

Spectral Distributions

n = 6 **n = 126**

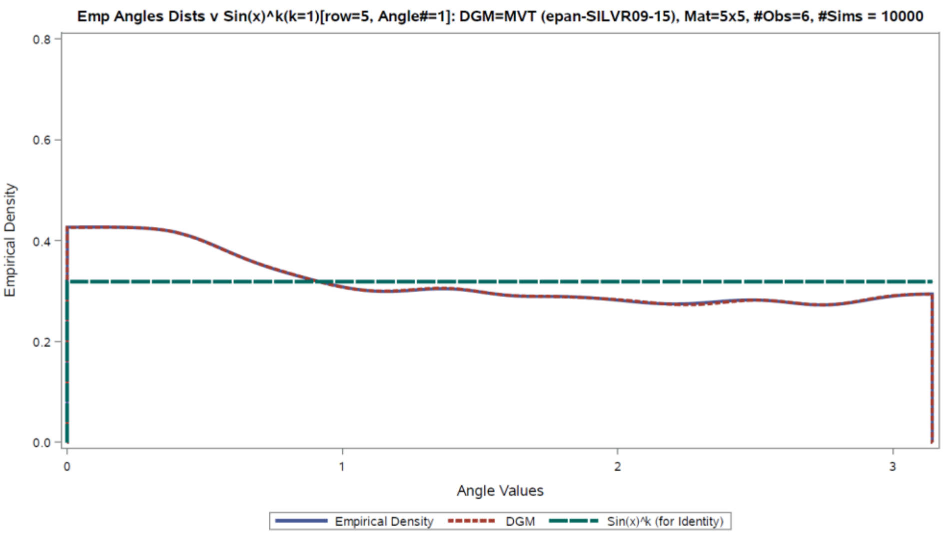




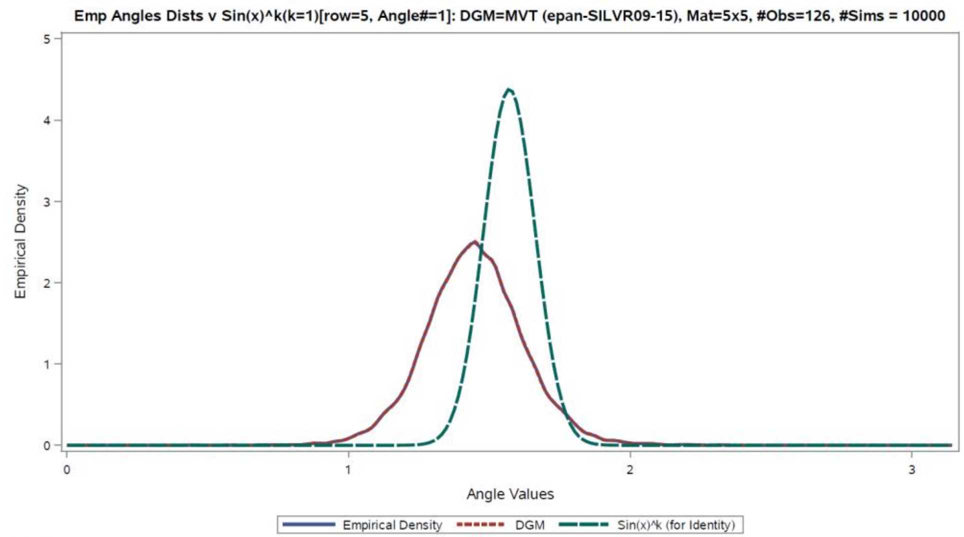
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case D

Angle Distributions

n = 6 **n = 126**



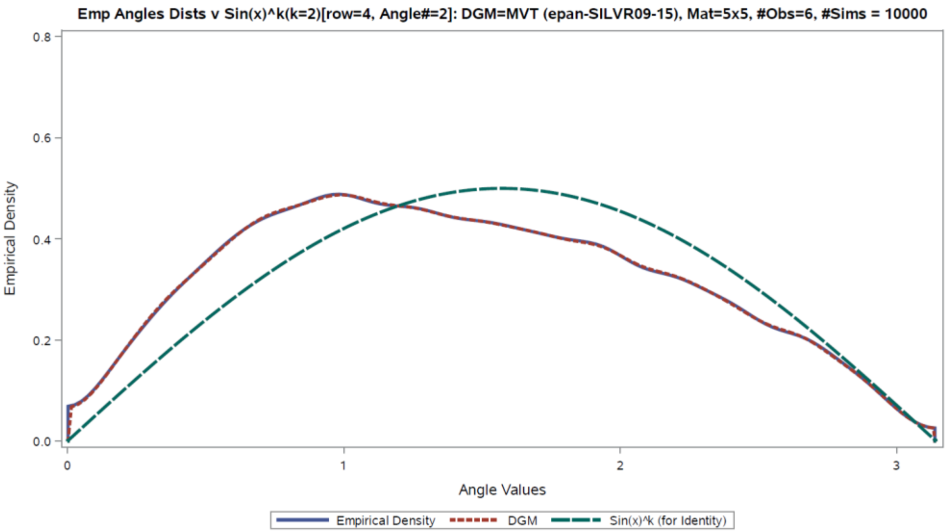
PRNG Seed = 12345
RunID = Dns5x5MVT



PRNG Seed = 12345
RunID = Dnb5x5MVT

Angle Distributions

n = 6 **n = 126**



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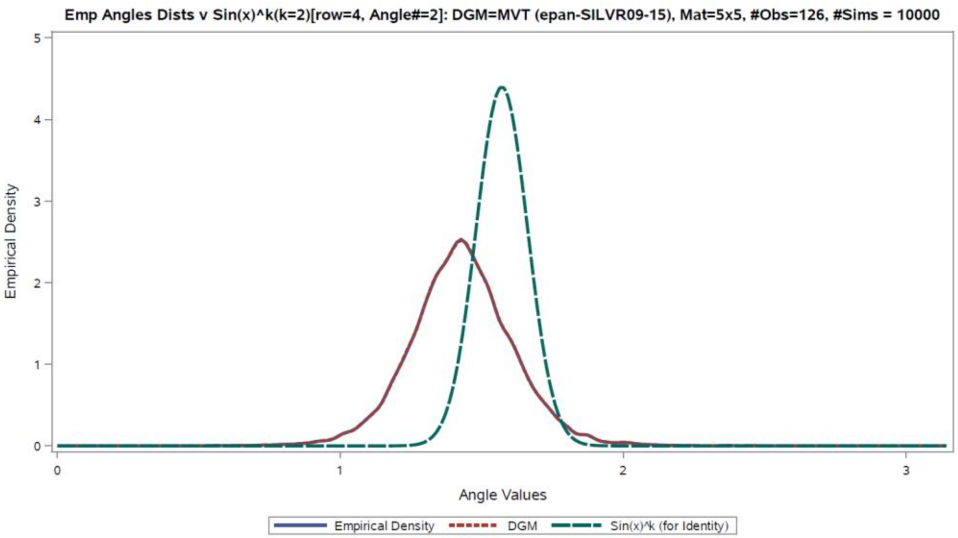
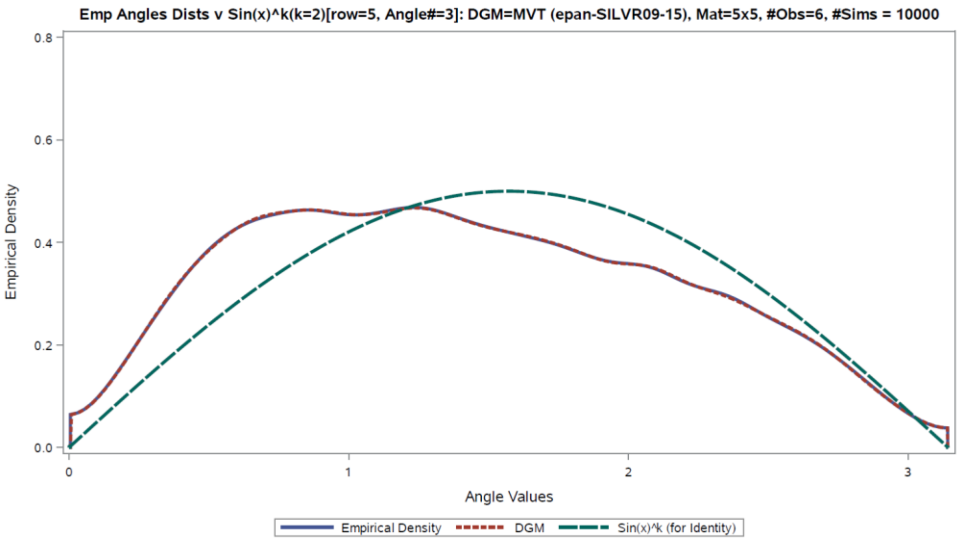
RunID = Dns5x5MVT



PRNG Seed = 12345
RunID = Dnb5x5MVT

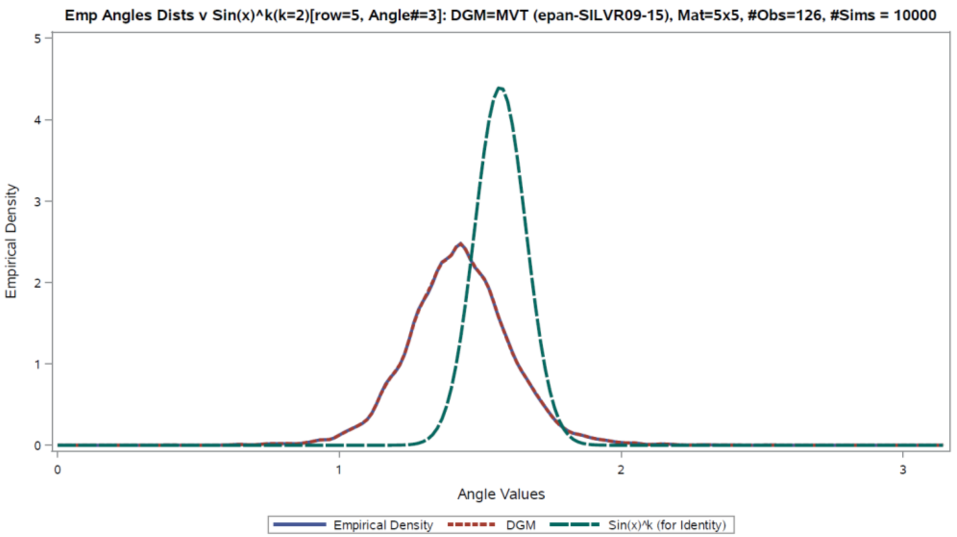
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Dns5x5MVT

n = 126

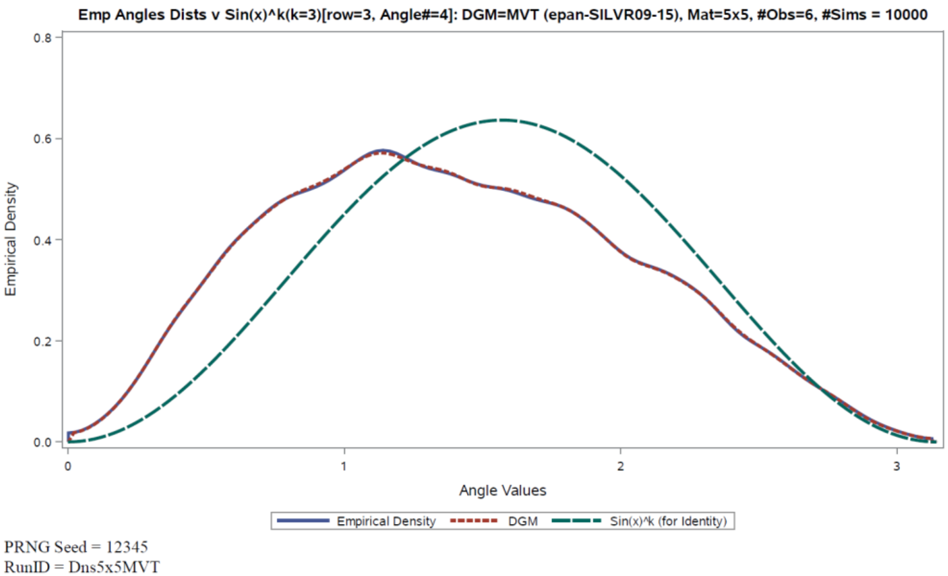


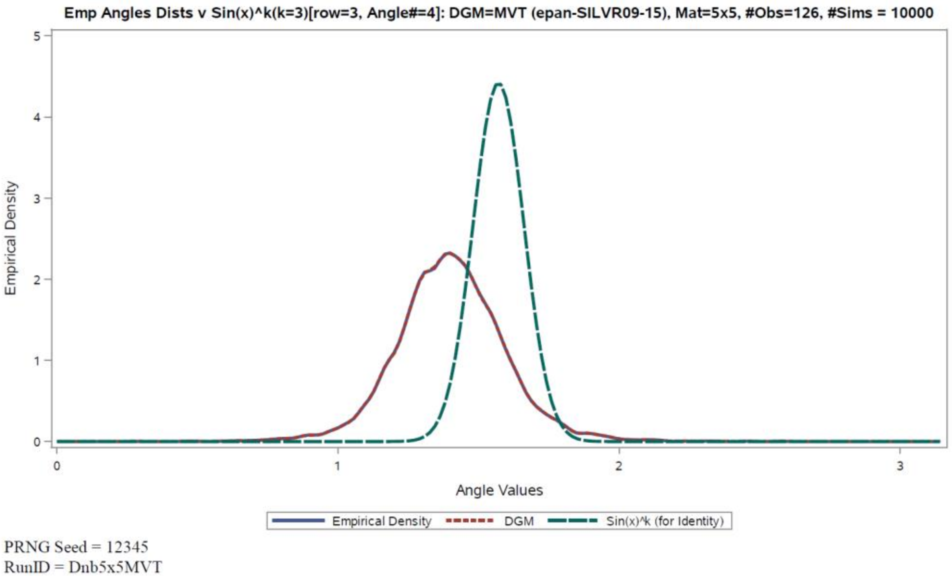
PRNG Seed = 12345
RunID = Dnb5x5MVT

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case D

Angle Distributions

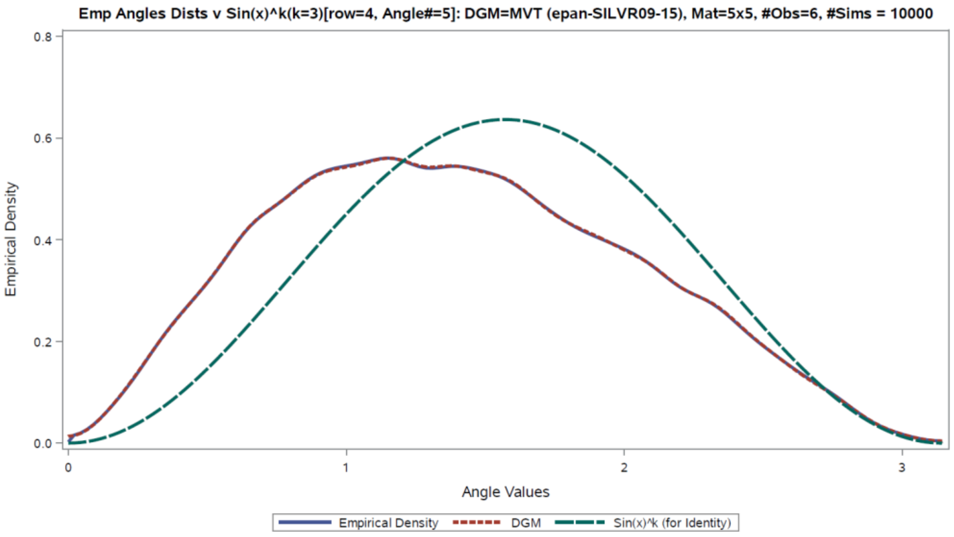
n = 6 **n = 126**





Angle Distributions

n = 6 **n = 126**



PRNG Seed = 12345
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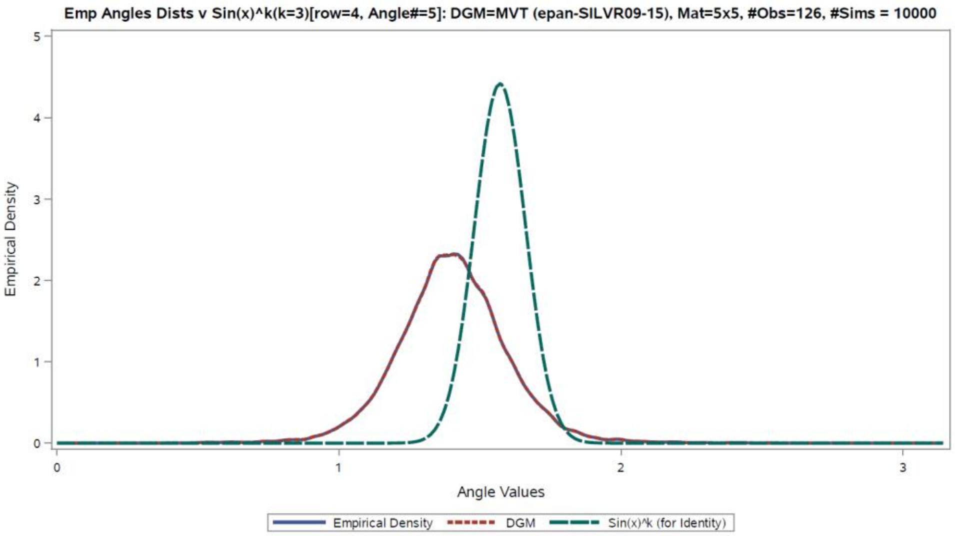
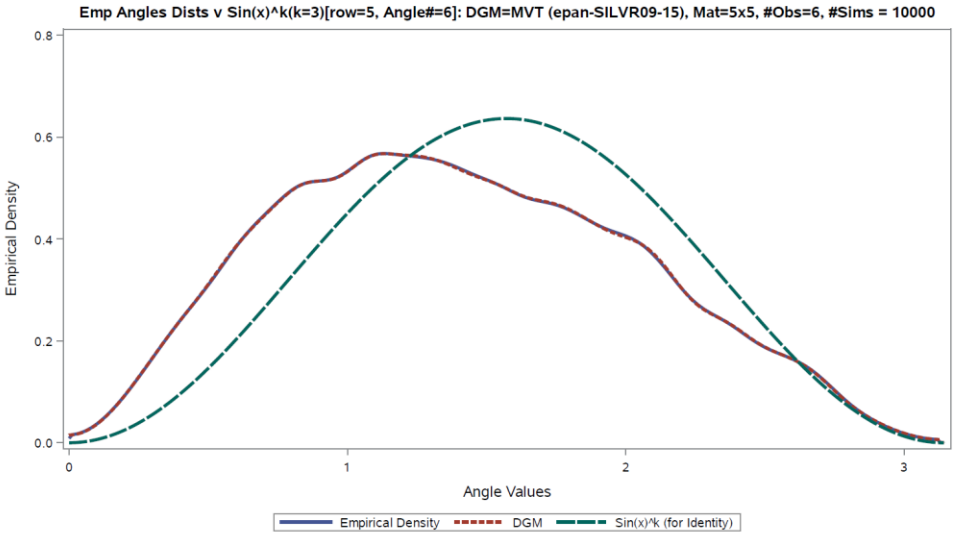


PRNG Seed = 12345
RunID = Dnb5x5MVT

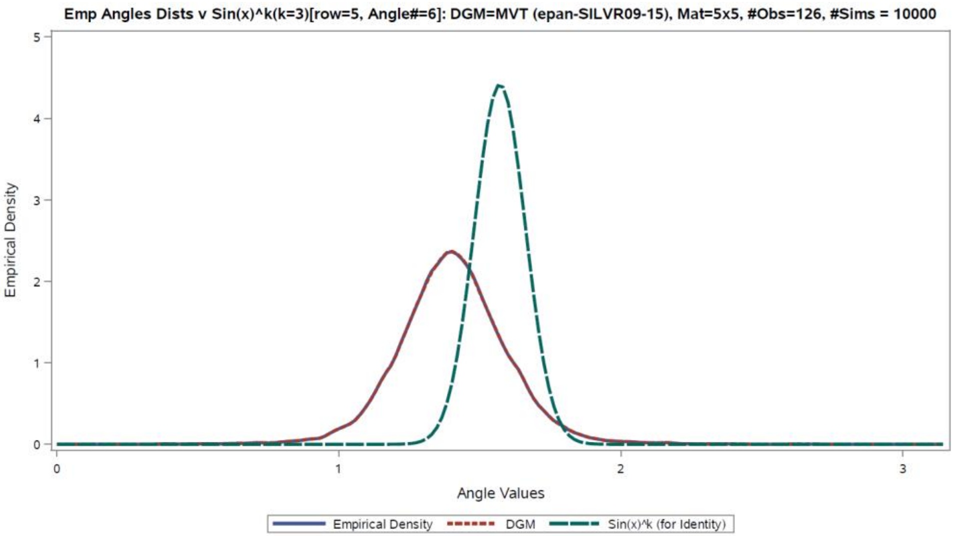
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Dns5x5MVT

n = 126

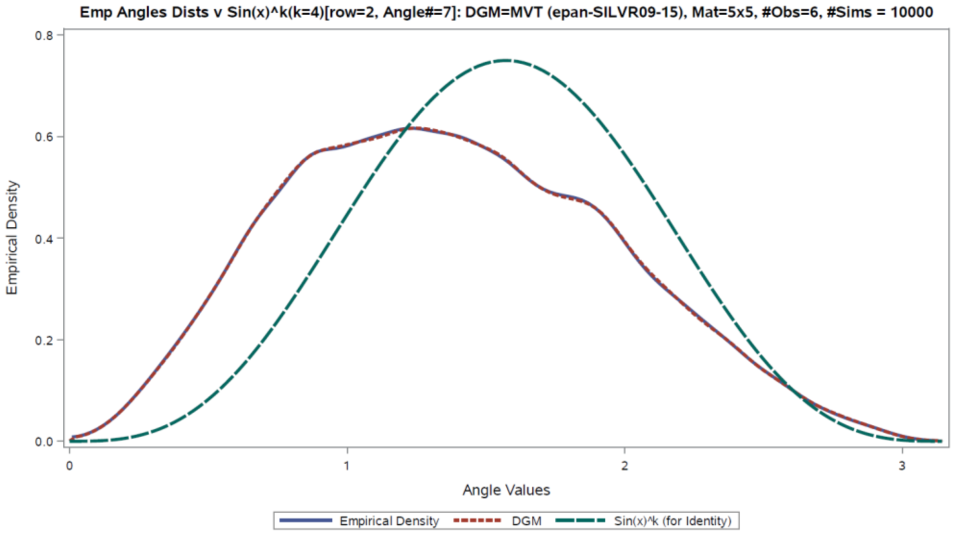


PRNG Seed = 12345
RunID = Dnb5x5MVT

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case D

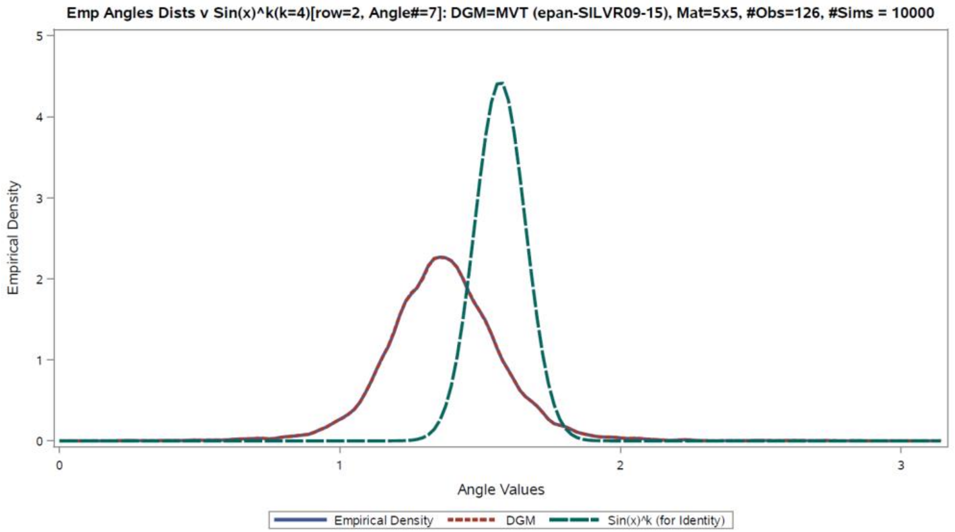
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Dns5x5MVT

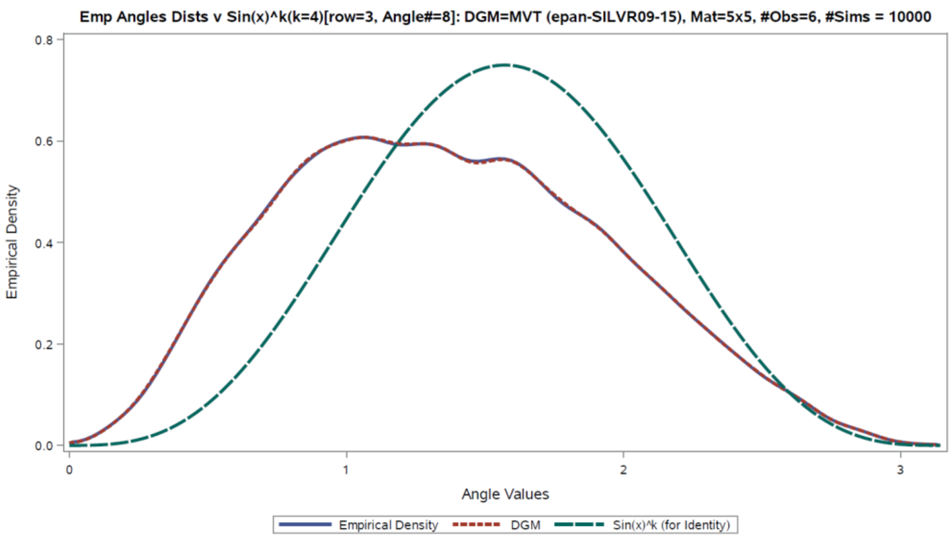
n = 126



PRNG Seed = 12345
RunID = Dnb5x5MVT

Angle Distributions

n = 6



PRNG Seed = 12345
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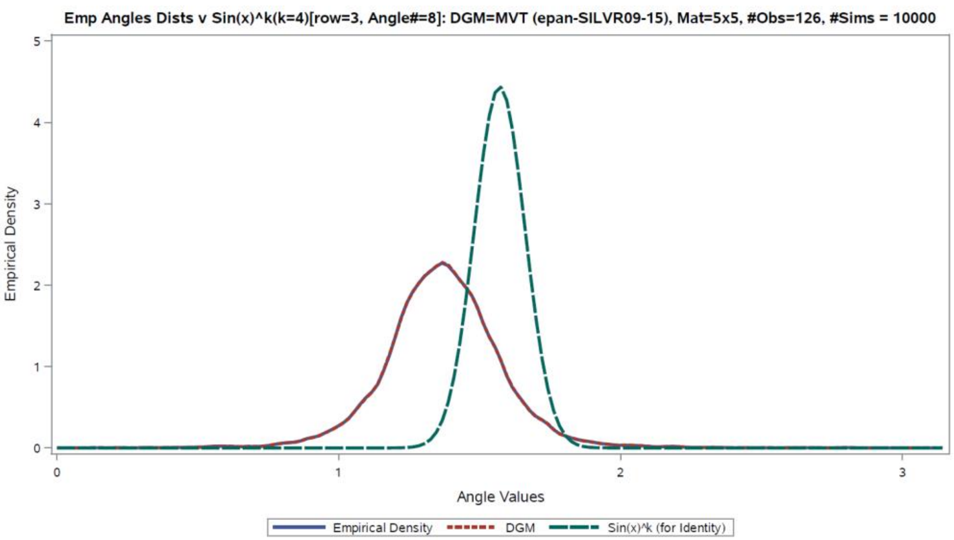
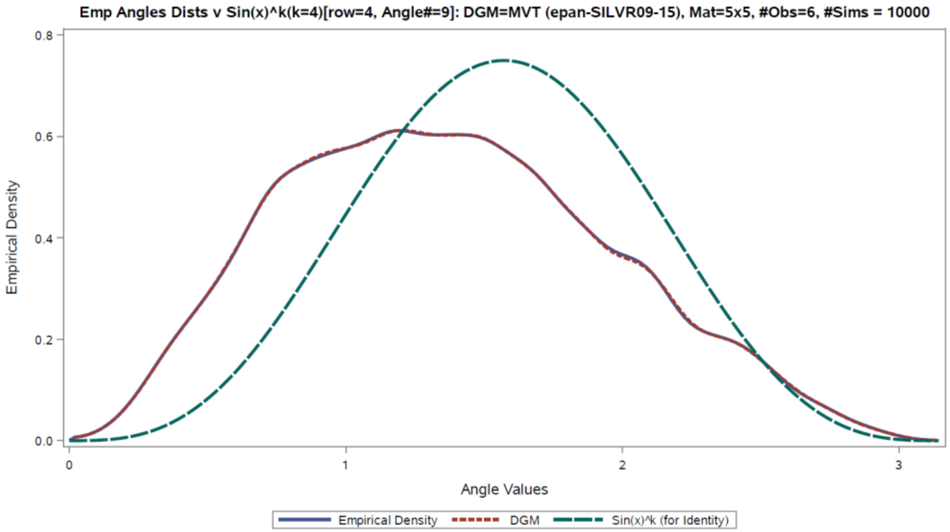
n = 126



PRNG Seed = 12345
RunID = Dnb5x5MVT

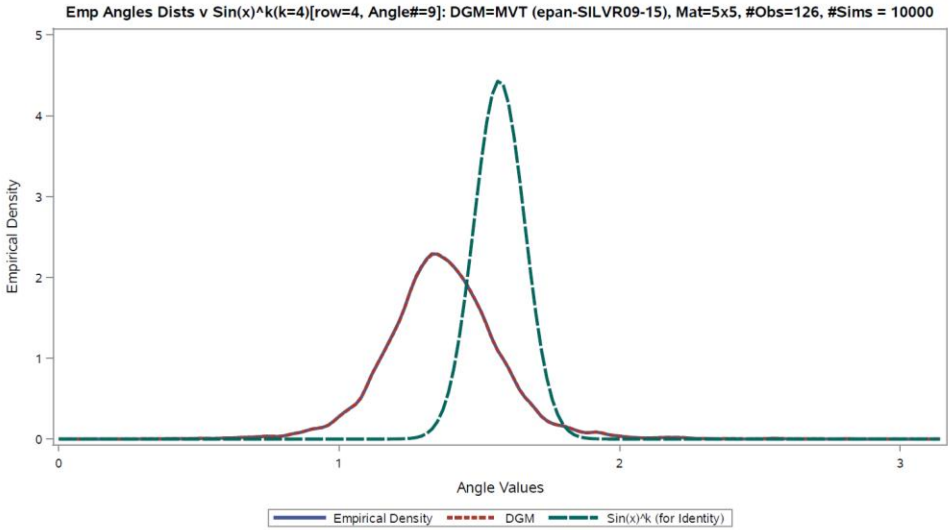
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Dns5x5MVT

n = 126

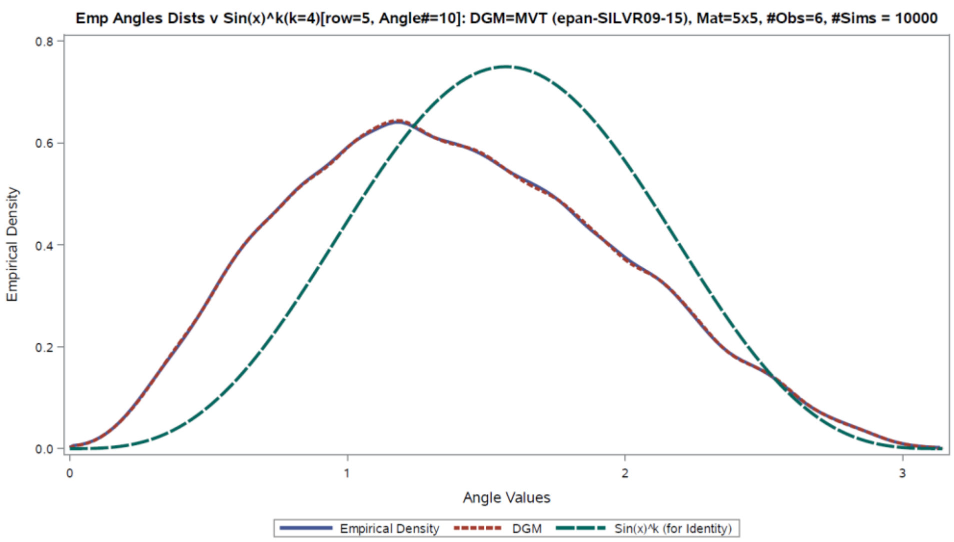


PRNG Seed = 12345
RunID = Dnb5x5MVT

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case D

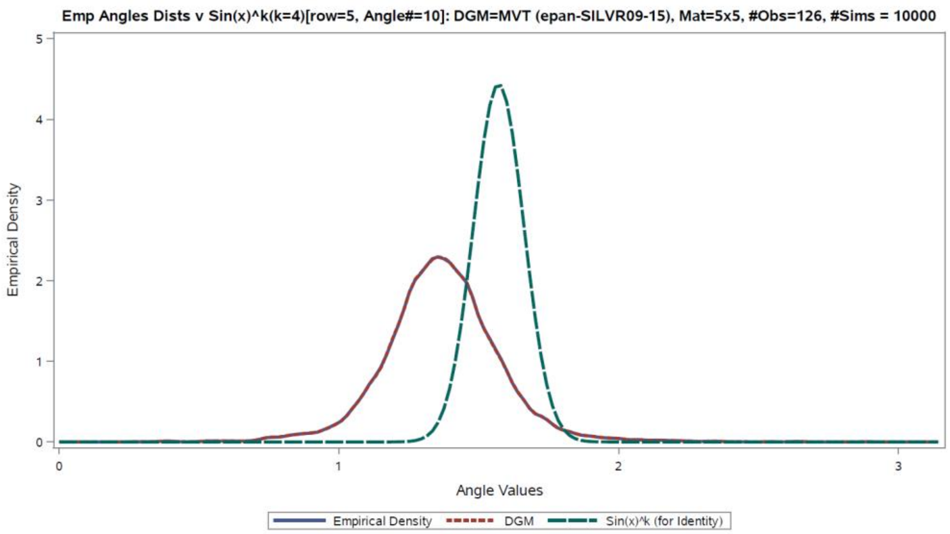
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Dns5x5MVT

n = 126



PRNG Seed = 12345
RunID = Dnb5x5MVT

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case D

Correlation Matrices to CDF Matrices

Correlation Matrices | **n=6** | **n=126**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.64	0.58	0.52	0.69	0.69		0.86	0.72	0.51	0.94	0.94
0	0.1	0.2	-0.1	-0.1		0.64	0.58	0.52	0.69	0.69		0.87	0.72	0.50	0.94	0.94
0	0.1	0.2	-0.1	-0.1		0.64	0.58	0.52	0.69	0.69		0.87	0.72	0.51	0.94	0.94
0	0.1	0.2	-0.1	-0.1		0.64	0.58	0.52	0.69	0.69		0.87	0.72	0.51	0.94	0.94
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.61	0.56	0.52	0.66	0.51		0.83	0.68	0.51	0.94	0.45
0	0.1	0.2	-0.1	0.2		0.61	0.57	0.53	0.67	0.51		0.83	0.68	0.51	0.93	0.45
0	0.1	0.2	-0.1	0.2		0.61	0.56	0.52	0.66	0.51		0.83	0.68	0.50	0.93	0.44
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.58	0.55	0.52	0.64	0.65		0.80	0.65	0.50	0.93	0.95
0	0.1	0.2	-0.1	-0.1		0.59	0.55	0.52	0.64	0.65		0.81	0.65	0.51	0.94	0.95
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.56	0.54	0.52	0.60	0.48		0.78	0.62	0.50	0.94	0.22
1	1	1	1	1												

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case D

CDF Matrices to Correlation Matrices

CDF Matrices | **n=6** | **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.23	0.38	0.06	0.46	-0.57		0.20	0.25	0.16	0.27	-0.04
0.5	0.4	0.6	0.35	0.9		0.24	0.40	0.06	0.47	-0.56		0.20	0.24	0.16	0.27	-0.04
0.5	0.4	0.6	0.35	0.9		0.23	0.38	0.07	0.46	-0.57		0.21	0.25	0.16	0.27	-0.03
0.5	0.4	0.6	0.35	0.9		0.23	0.38	0.06	0.45	-0.58		0.20	0.25	0.16	0.27	-0.03
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.25	0.48	0.02	0.22	0.04		0.21	0.26	0.15	0.19	0.02
0.5	0.4	0.6	0.6	0.8		0.25	0.47	0.03	0.23	0.05		0.20	0.26	0.15	0.19	0.03
0.5	0.4	0.6	0.6	0.8		0.25	0.47	0.02	0.22	0.05		0.20	0.26	0.15	0.19	0.02
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.27	0.57	-0.03	0.53	0.76		0.20	0.27	0.14	0.25	0.23
0.5	0.4	0.6	0.4	0.3		0.27	0.56	-0.03	0.53	0.77		0.21	0.27	0.14	0.26	0.23
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.29	0.69	-0.14	0.36	0.27		0.20	0.28	0.13	0.21	-0.10
						1	1	1	1	1		1	1	1	1	1

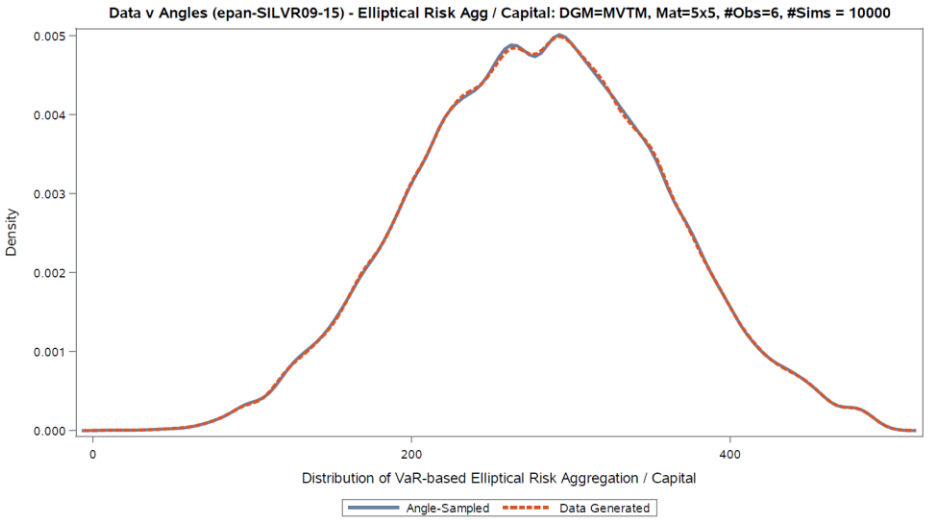
CDF %Shift Matrices to Correlation Matrices

CDF %Shift Matrices — **n=6** — **n=126**

					lnMatPr	-2.395	-2.071	-7.602	-6.304	-14.832		-2.297	-2.167	-7.196	-6.675	-13.439
					FNorm	1.175	1.010	2.478	1.800	2.910		0.253	0.237	0.679	0.625	1.058
					Rnk_lnMat	2	1	4	3	5		2	1	4	3	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.34	0.01	0.58	-0.26	0.65		0.24	0.15	0.31	0.08	0.34
20	-20	50	-50	60		0.35	0.01	0.59	-0.25	0.66		0.24	0.15	0.31	0.08	0.34
20	-20	50	-50	60		0.34	0.02	0.58	-0.25	0.65		0.24	0.15	0.31	0.08	0.34
20	-20	50	-50	60		0.34	0.01	0.56	-0.26	0.65		0.24	0.16	0.31	0.08	0.34
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.42	-0.03	0.73	-0.24	0.86		0.25	0.14	0.35	0.06	0.42
20	-20	50	-50	70		0.42	-0.02	0.73	-0.24	0.86		0.25	0.14	0.35	0.06	0.42
20	-20	50	-50	70		0.41	-0.03	0.72	-0.24	0.85		0.25	0.14	0.35	0.05	0.42
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.50	-0.08	0.85	-0.19	0.97		0.26	0.13	0.37	0.04	0.49
20	-20	50	-50	80		0.49	-0.09	0.85	-0.21	0.97		0.26	0.14	0.38	0.04	0.50
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.62	-0.20	0.95	-0.12	1.00		0.27	0.13	0.40	0.02	0.59
						1	1	1	1	1		1	1	1	1	1

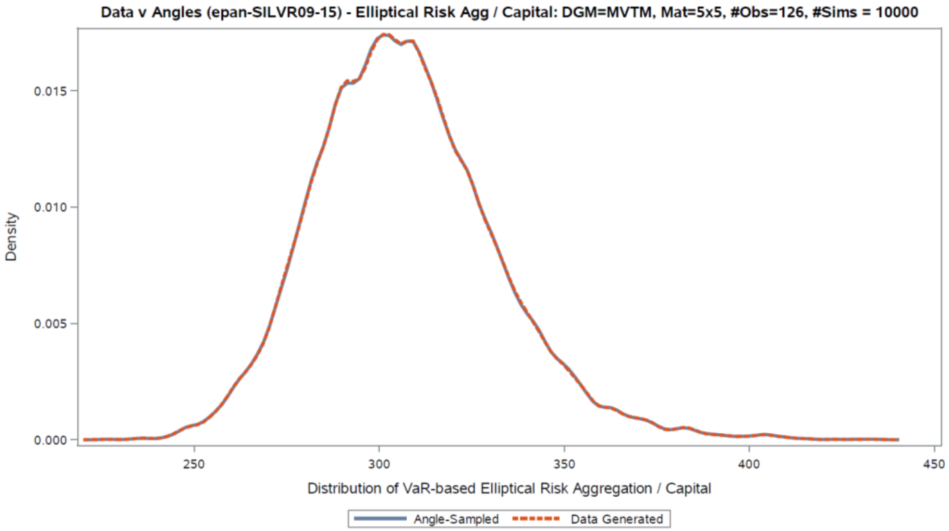
Elliptical Capital

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

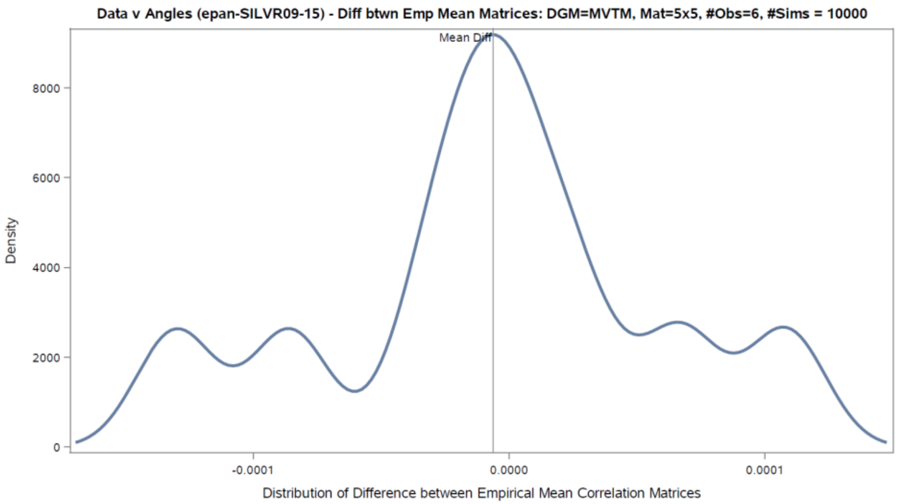
n = 126



PRNG Seed = 12345
RunID = Eb5x5MVTM

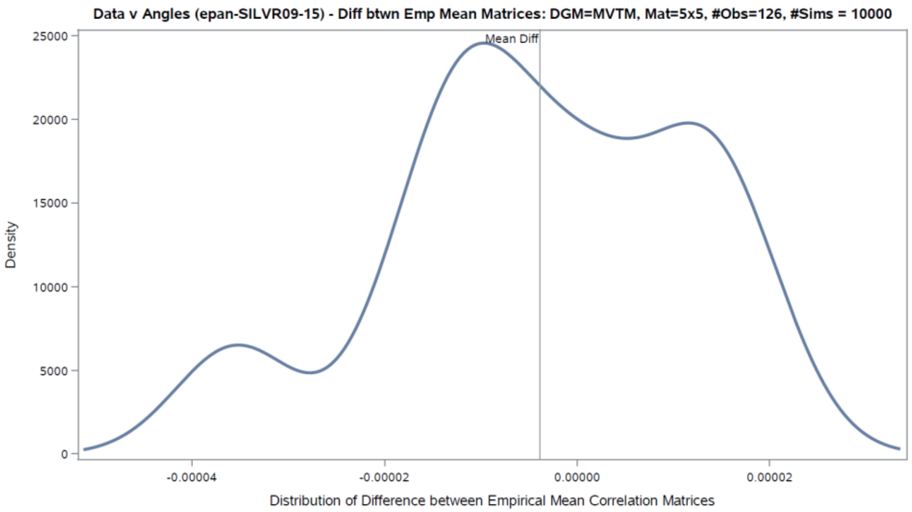
Difference Between Mean Empirical Matrices

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

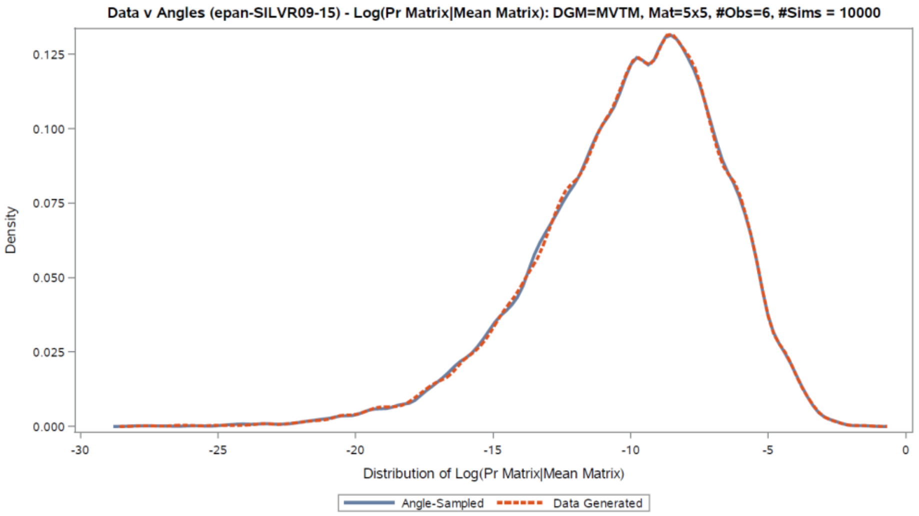
n = 126



PRNG Seed = 12345
RunID = Eb5x5MVTM

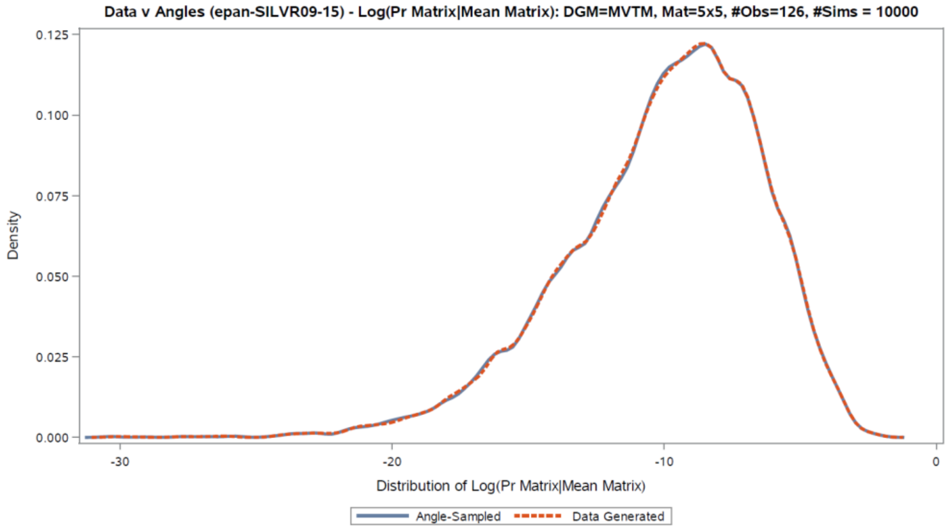
LNP

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

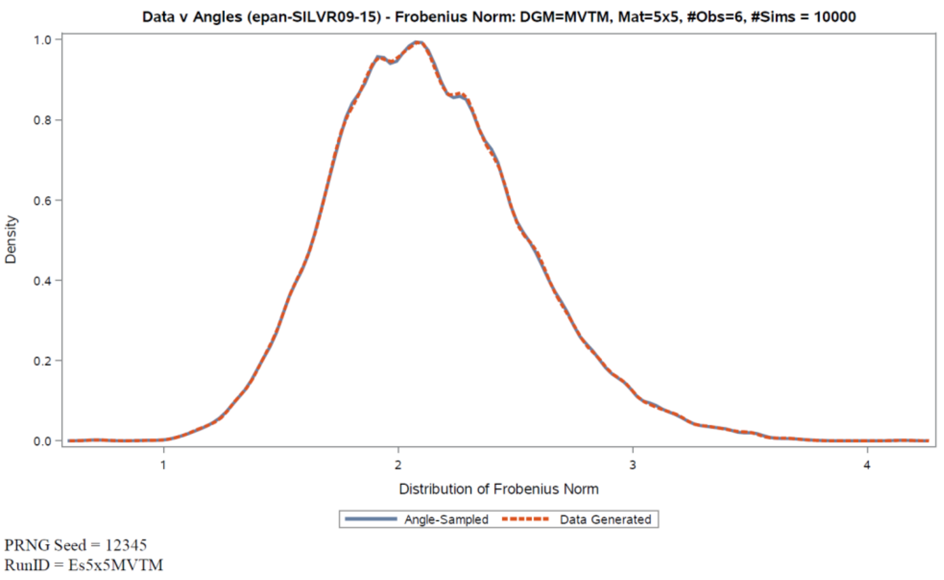
n = 126



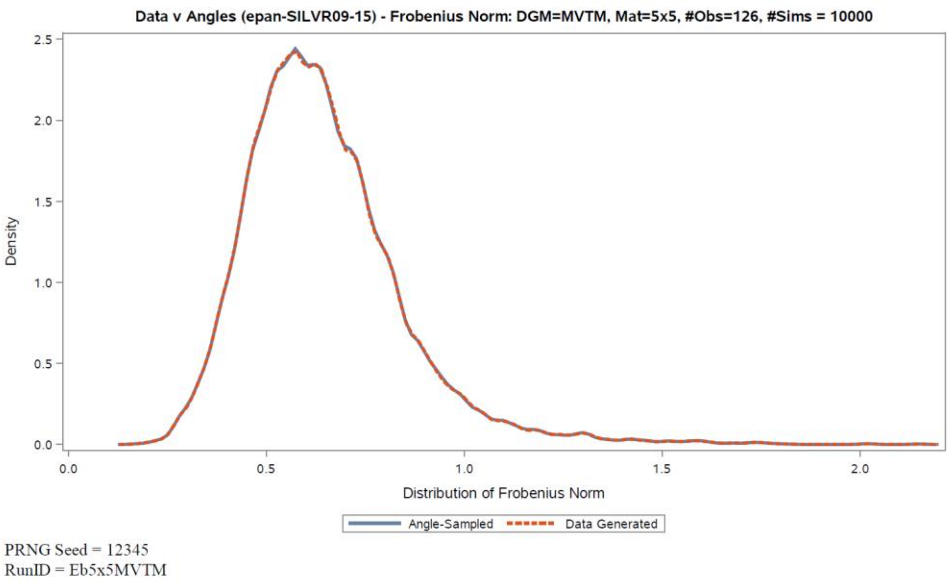
PRNG Seed = 12345
RunID = Eb5x5MVTM

Euclidian/Frobenius Norm

n = 6



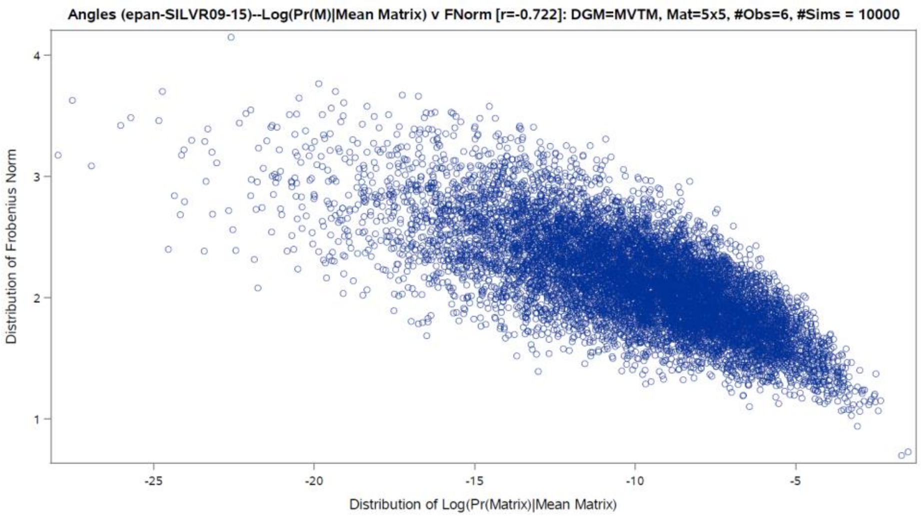
n = 126



XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

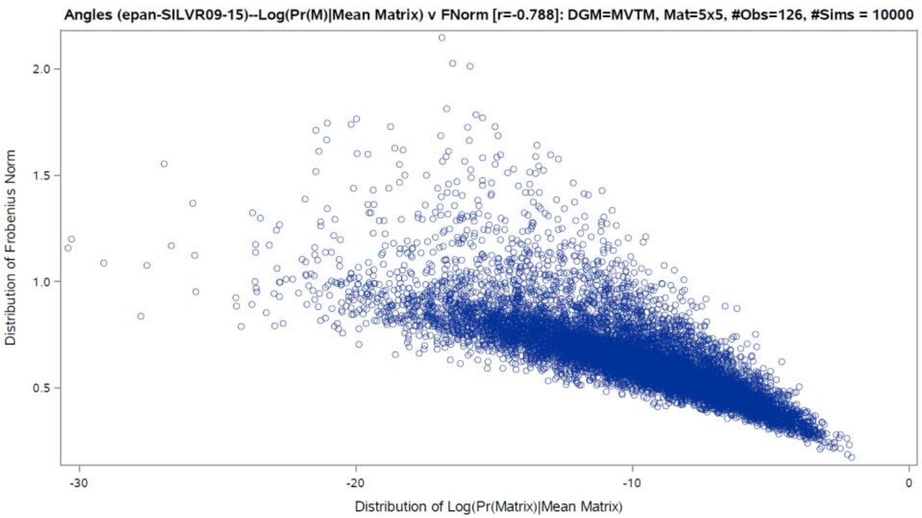
LNP v Euclidian/Frobenius Norm

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

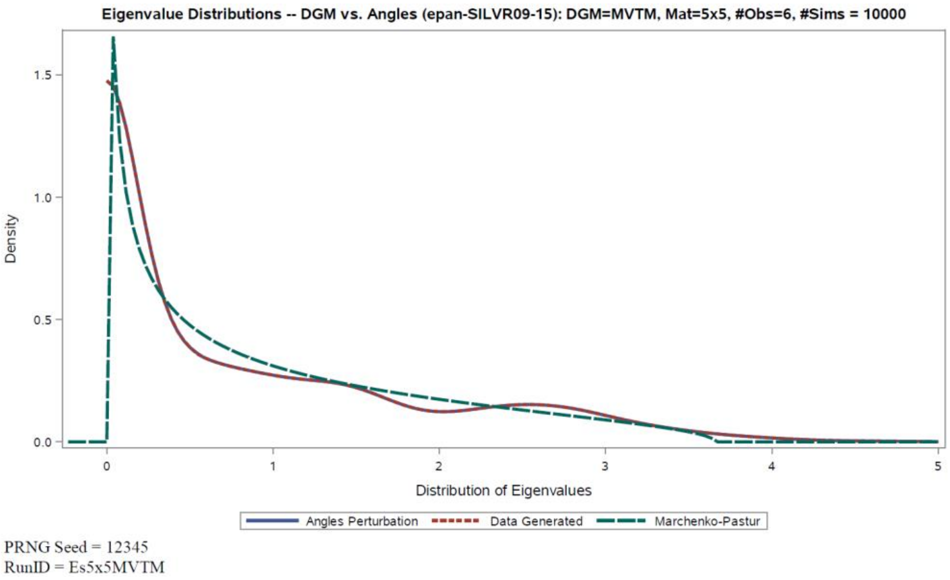
n = 126



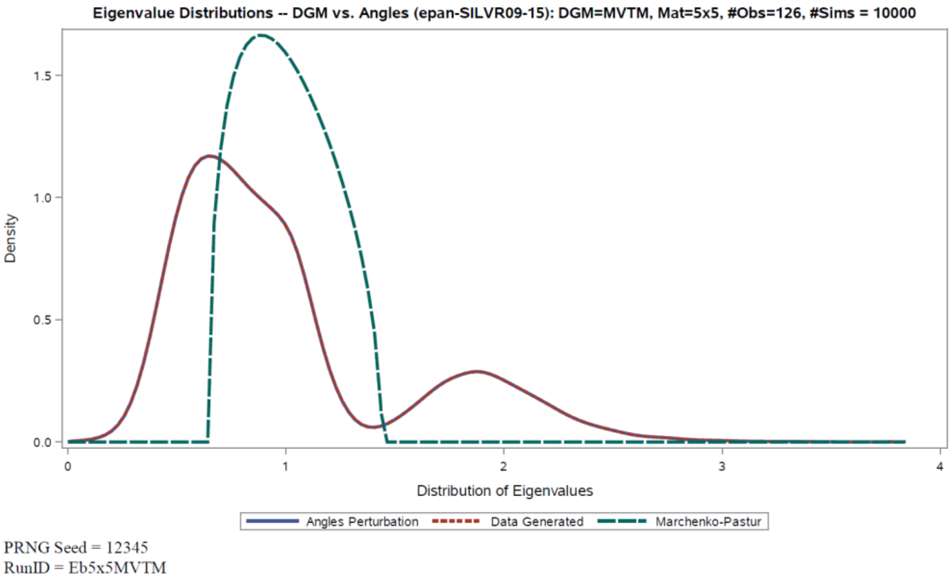
PRNG Seed = 12345
RunID = Eb5x5MVTM

Spectral Distributions

n = 6

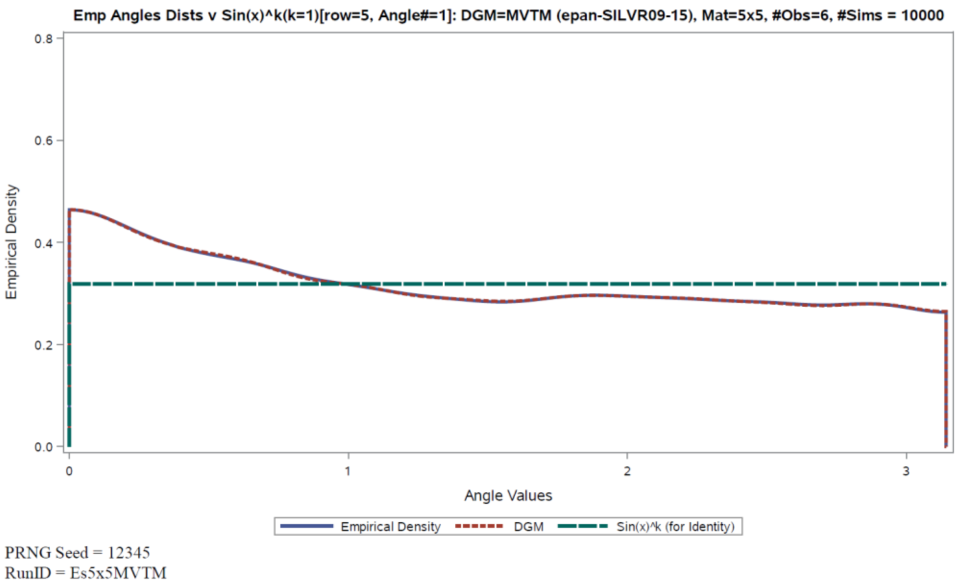


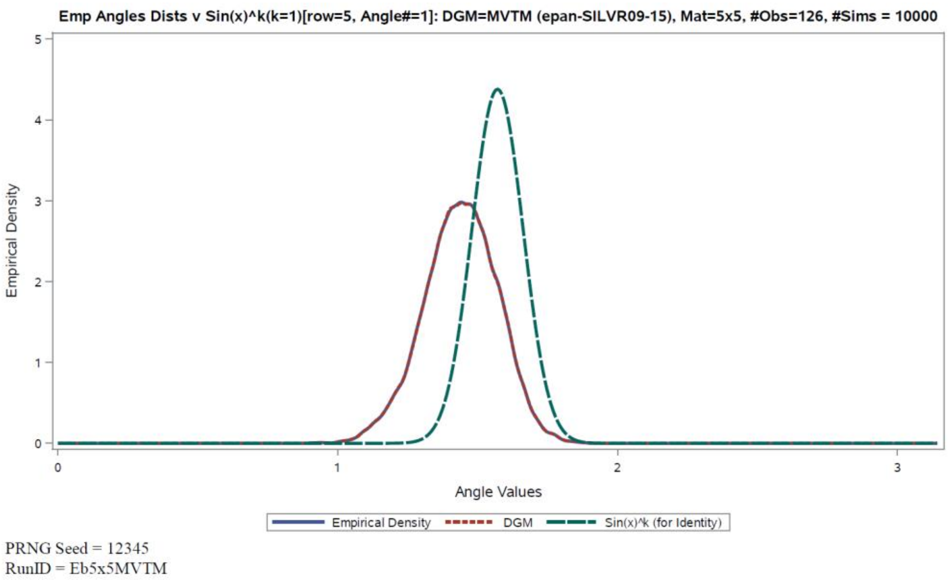
n = 126



Angle Distributions

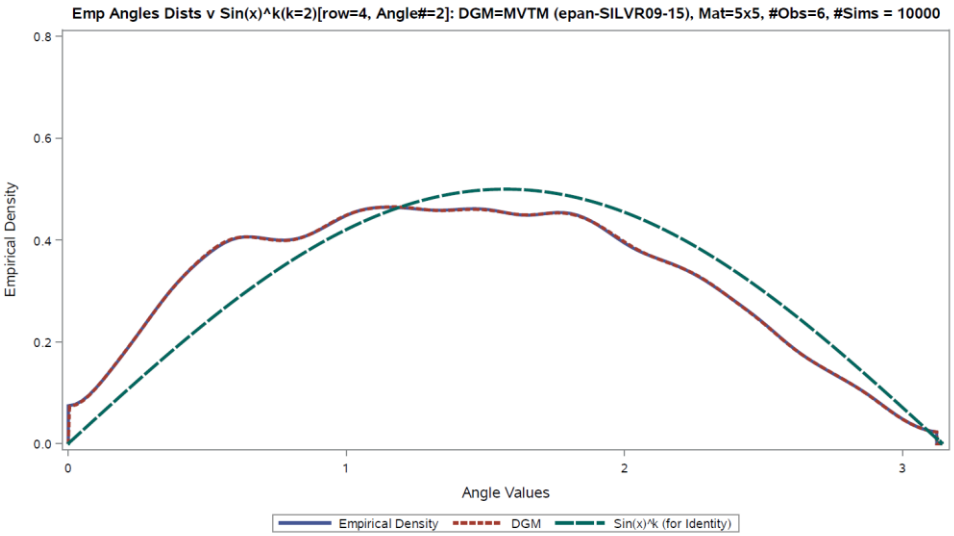
n = 6 **n = 126**





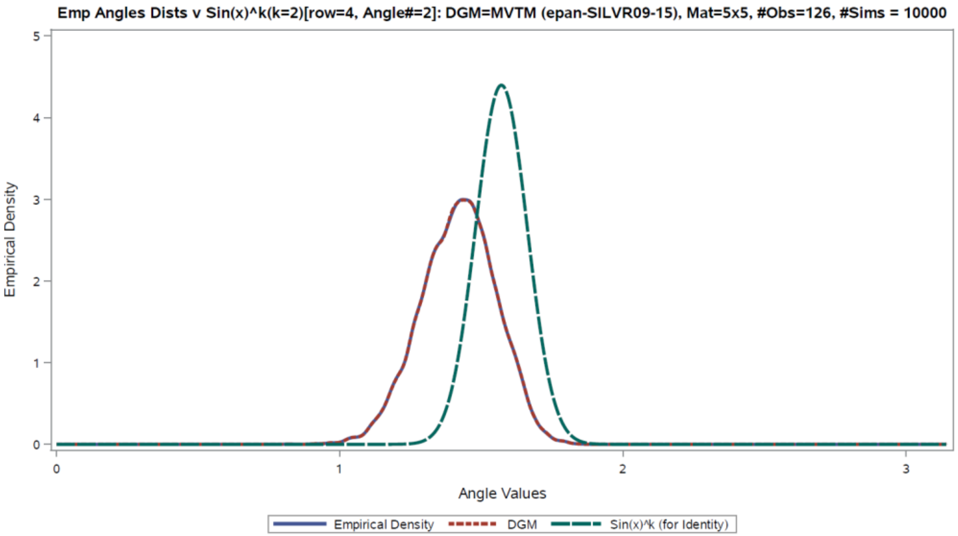
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

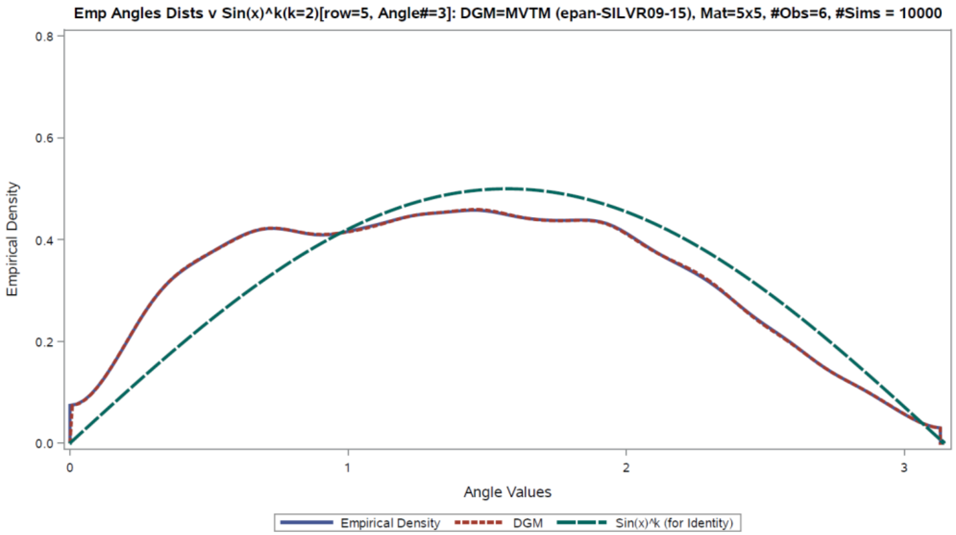
n = 126



PRNG Seed = 12345
RunID = Eb5x5MVTM

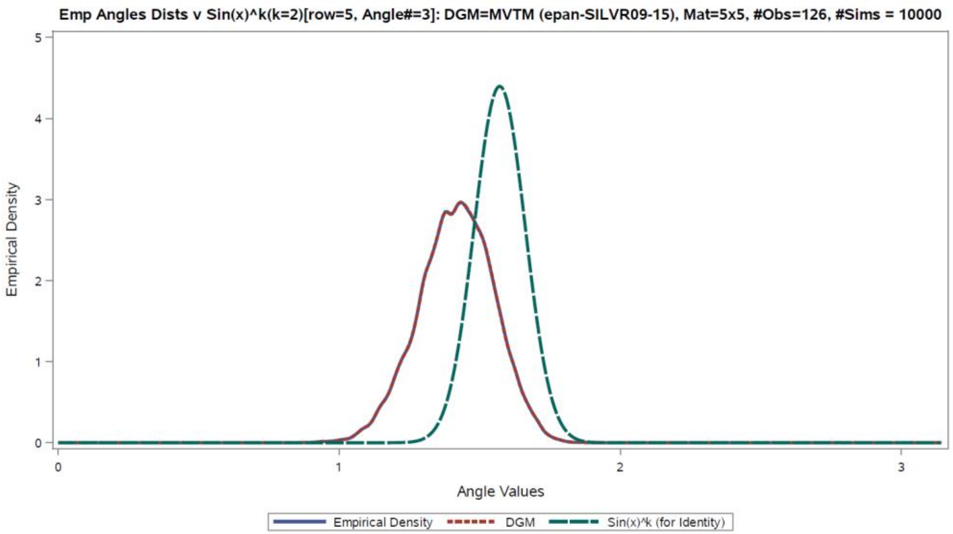
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

n = 126

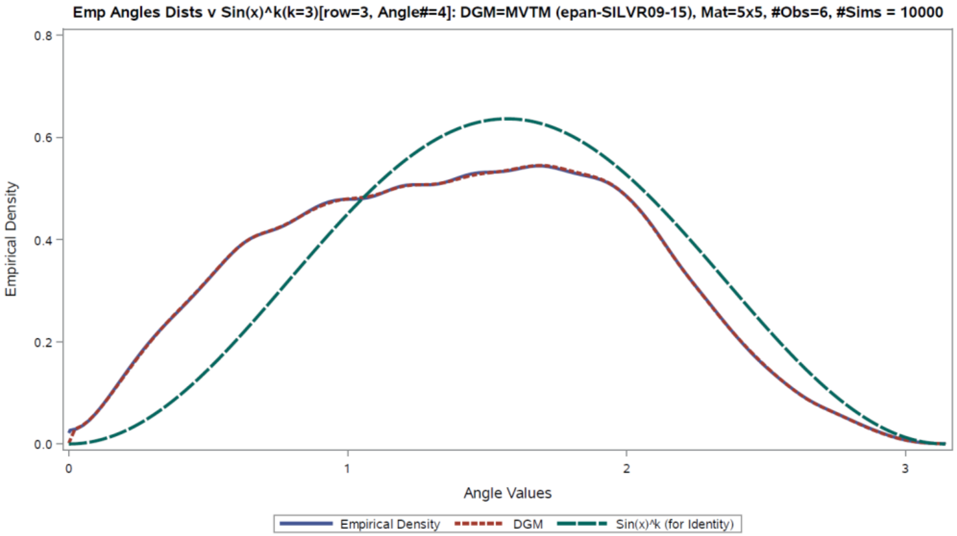


PRNG Seed = 12345
RunID = Eb5x5MVTM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

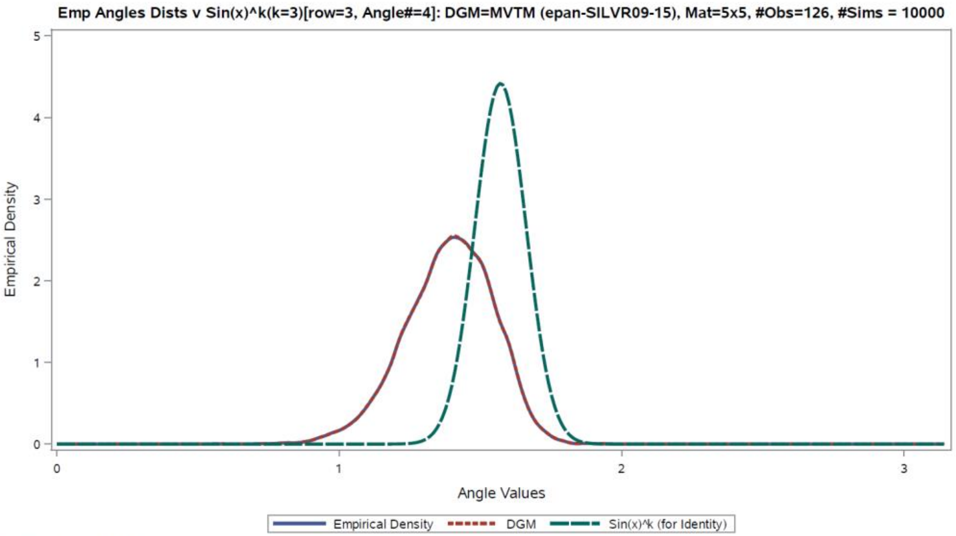
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

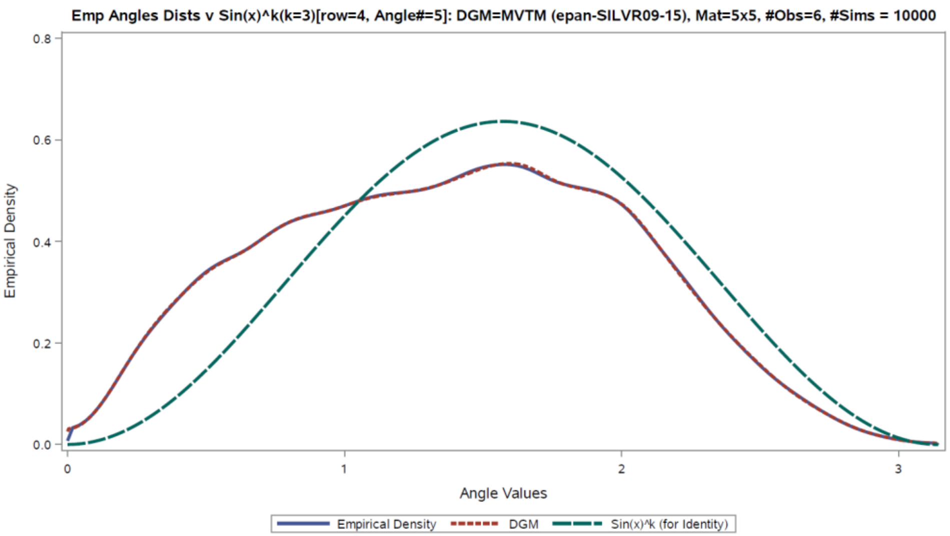
n = 126



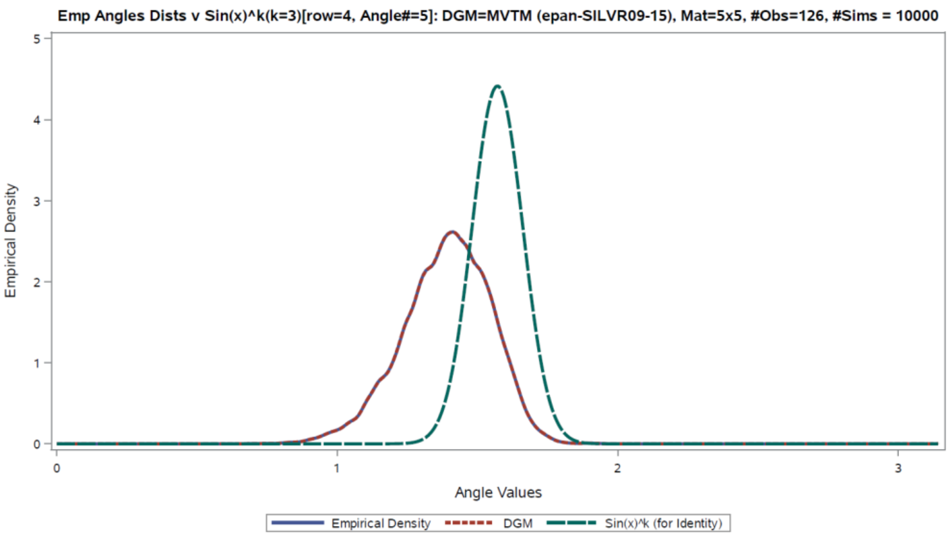
PRNG Seed = 12345
RunID = Eb5x5MVTM

Angle Distributions

n = 6 **n = 126**



PRNG Seed = 12345
RunID = Es5x5MVTM

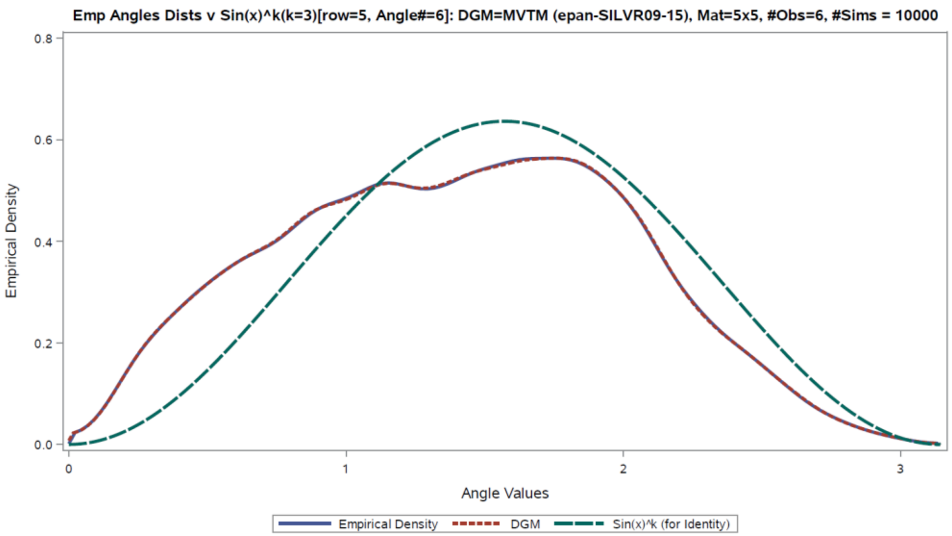


PRNG Seed = 12345
RunID = Eb5x5MVTM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

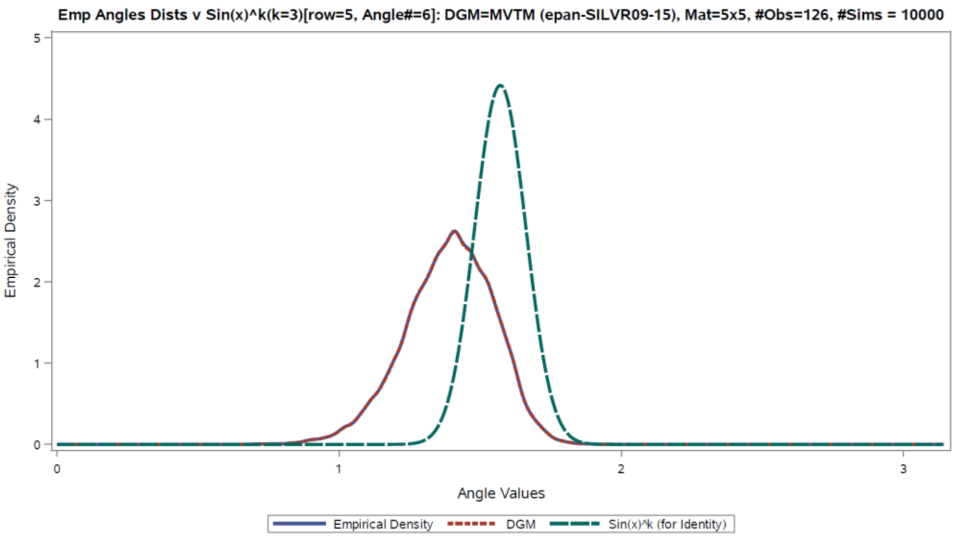
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

n = 126

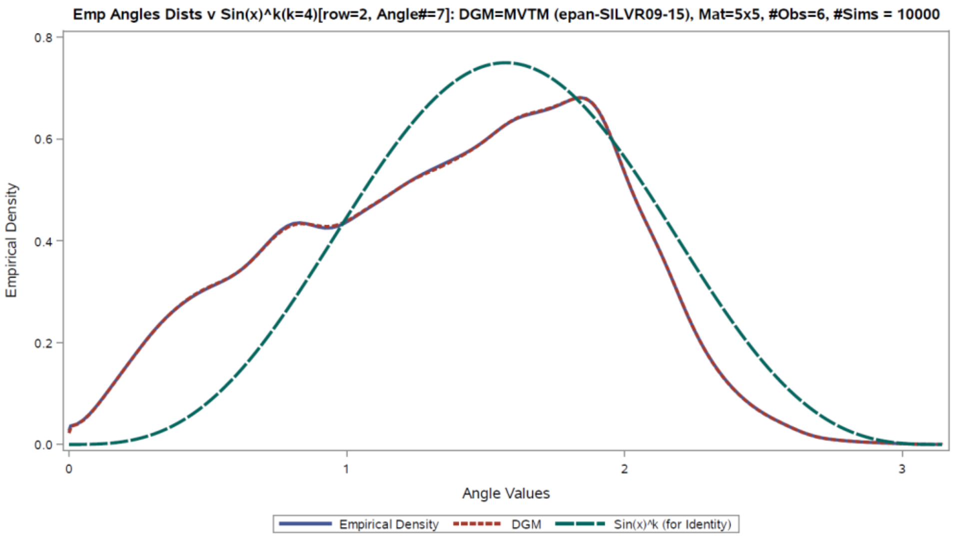


PRNG Seed = 12345
RunID = Eb5x5MVTM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

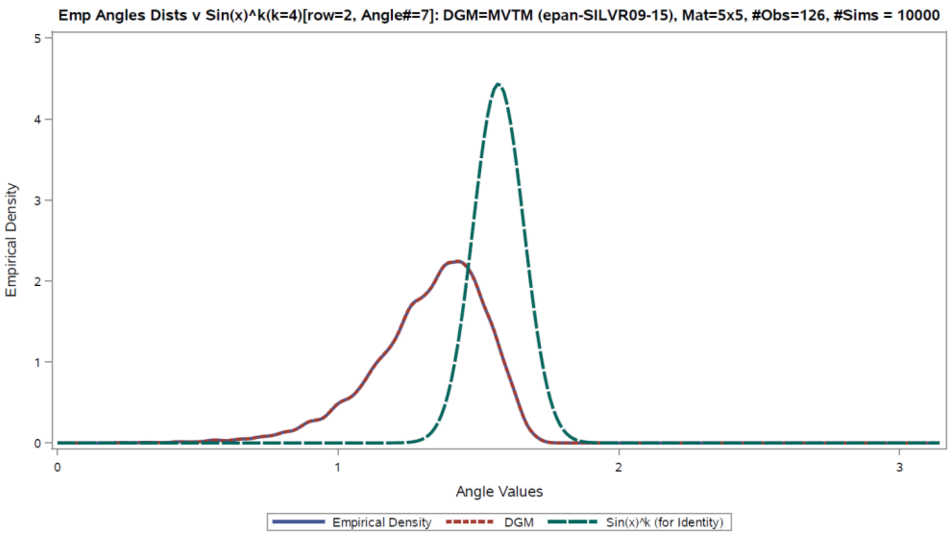
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

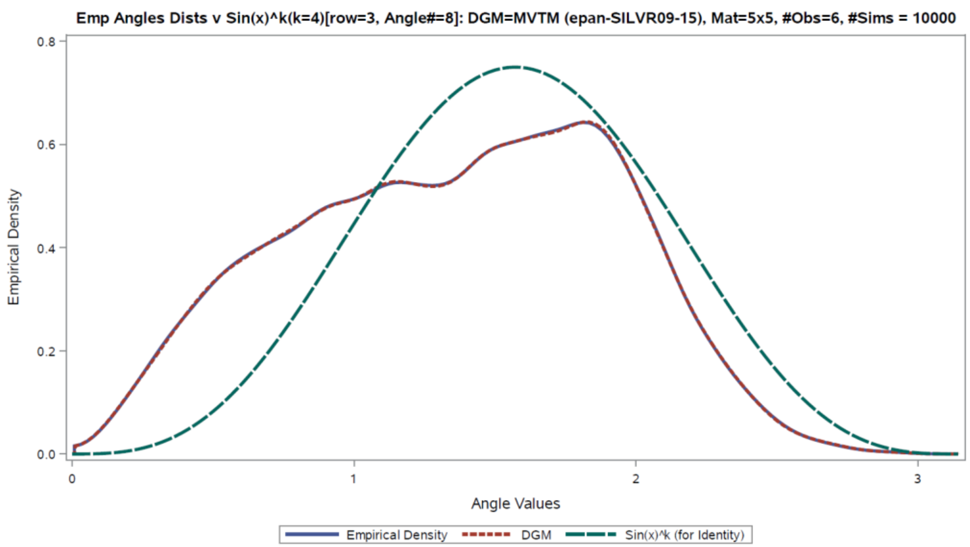
n = 126



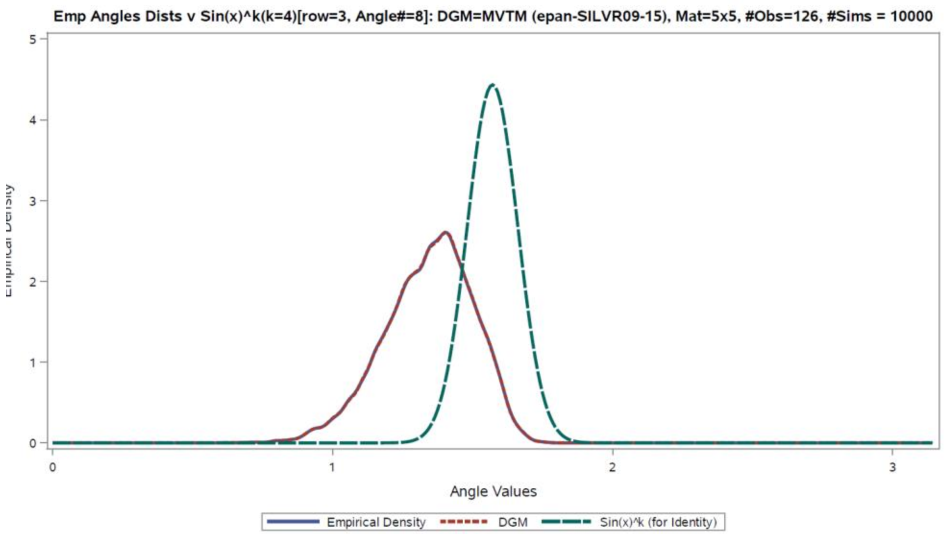
PRNG Seed = 12345
RunID = Eb5x5MVTM

Angle Distributions

n = 6 **n = 126**



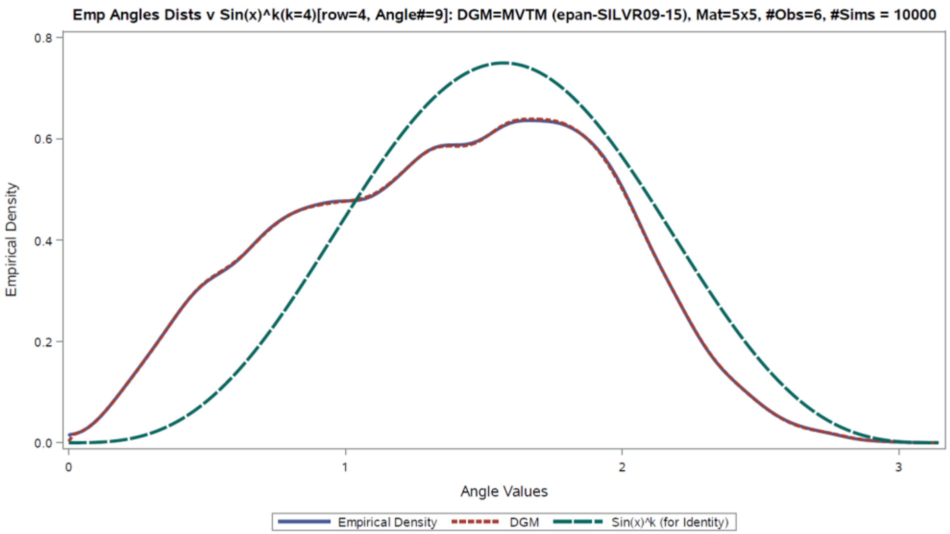
PRNG Seed = 12345
RunID = Es5x5MVTM



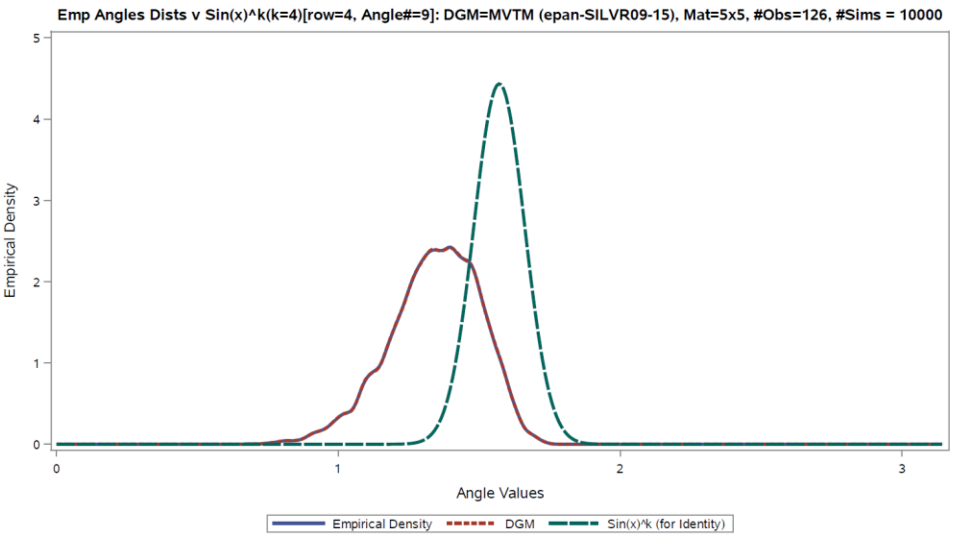
ʼRNG Seed = 12345
ℓunID = Eb5x5MVTM

Angle Distributions

n = 6 **n = 126**



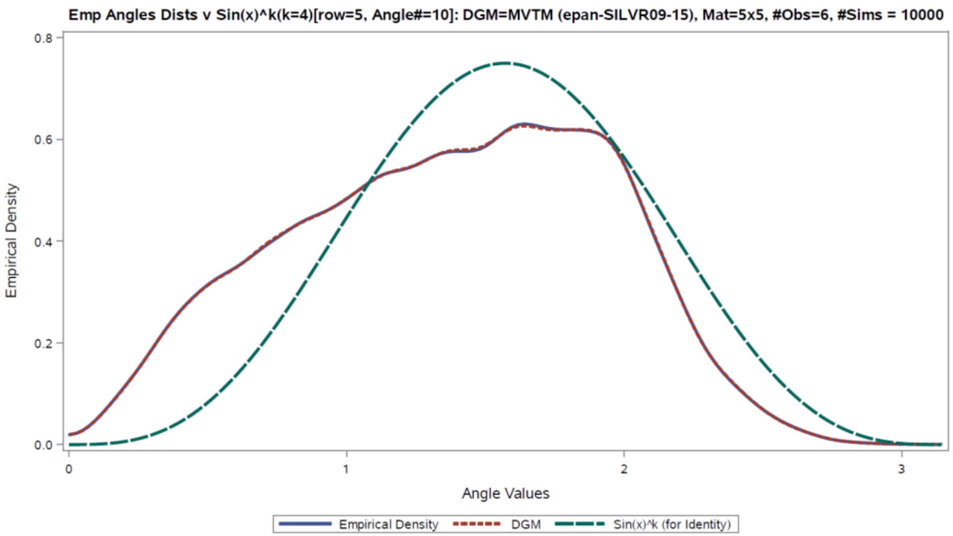
PRNG Seed = 12345
RunID = Es5x5MVTM



PRNG Seed = 12345
RunID = Eb5x5MVTM

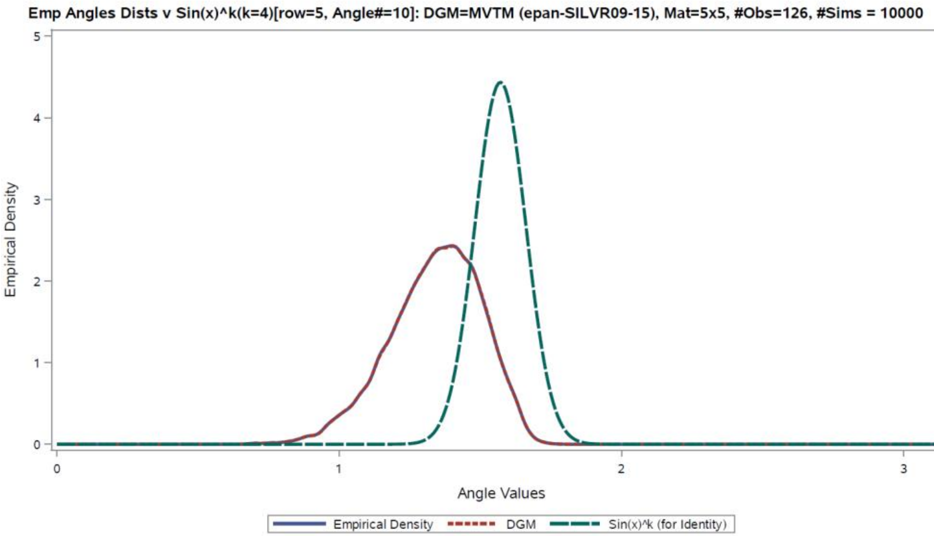
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Es5x5MVTM

n = 126



PRNG Seed = 12345
RunID = Eb5x5MVTM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

Correlation Matrices to CDF Matrices

Correlation Matrices | **n=6** | **n=126**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.59	0.53	0.47	0.65	0.65		0.926	0.753	0.528	0.995	0.995
0	0.1	0.2	-0.1	-0.1		0.60	0.54	0.48	0.66	0.66		0.940	0.781	0.531	0.995	0.995
0	0.1	0.2	-0.1	-0.1		0.60	0.54	0.48	0.66	0.66		0.940	0.773	0.536	0.994	0.994
0	0.1	0.2	-0.1	-0.1		0.60	0.54	0.48	0.66	0.66		0.939	0.774	0.537	0.996	0.996
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.58	0.53	0.50	0.64	0.48		0.882	0.702	0.515	0.983	0.448
0	0.1	0.2	-0.1	0.2		0.59	0.54	0.50	0.65	0.48		0.886	0.706	0.512	0.984	0.445
0	0.1	0.2	-0.1	0.2		0.58	0.53	0.49	0.64	0.48		0.881	0.702	0.513	0.981	0.447
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.58	0.54	0.51	0.63	0.64		0.875	0.690	0.511	0.989	0.994
0	0.1	0.2	-0.1	-0.1		0.57	0.54	0.51	0.63	0.64		0.881	0.694	0.522	0.987	0.995
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.55	0.53	0.52	0.59	0.48		0.838	0.654	0.510	0.987	0.176
1	1	1	1	1												

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

CDF Matrices to Correlation Matrices

CDF Matrices **n=6** **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.143	0.322	-0.023	0.413	-0.476		0.21	0.26	0.17	0.29	0.02
0.5	0.4	0.6	0.35	0.9		0.167	0.357	0.001	0.439	-0.481		0.21	0.25	0.17	0.28	0.03
0.5	0.4	0.6	0.35	0.9		0.166	0.331	-0.002	0.420	-0.483		0.21	0.26	0.17	0.28	0.03
0.5	0.4	0.6	0.35	0.9		0.162	0.332	-0.006	0.416	-0.483		0.21	0.26	0.17	0.28	0.03
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.176	0.422	-0.031	0.156	-0.073		0.21	0.26	0.16	0.20	0.05
0.5	0.4	0.6	0.6	0.8		0.184	0.422	-0.021	0.156	-0.073		0.21	0.26	0.16	0.20	0.05
0.5	0.4	0.6	0.6	0.8		0.169	0.414	-0.035	0.142	-0.065		0.21	0.26	0.16	0.20	0.05
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.204	0.511	-0.055	0.486	0.715		0.21	0.27	0.16	0.26	0.22
0.5	0.4	0.6	0.4	0.3		0.202	0.513	-0.060	0.488	0.725		0.21	0.27	0.16	0.26	0.22
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.247	0.660	-0.162	0.297	0.129		0.21	0.28	0.15	0.22	-0.02
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case E

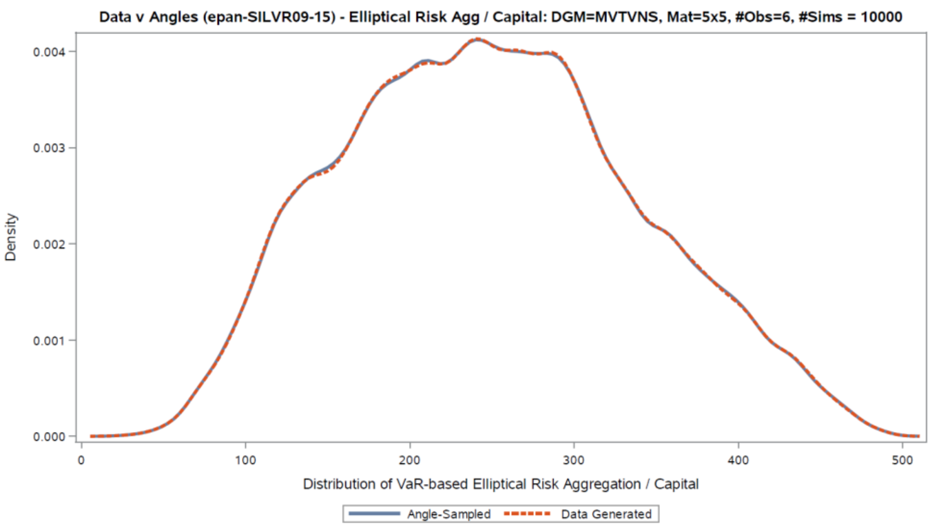
CDF %Shift Matrices to Correlation Matrices

CDF %Shift Matrices | **n=6** | **n=126**

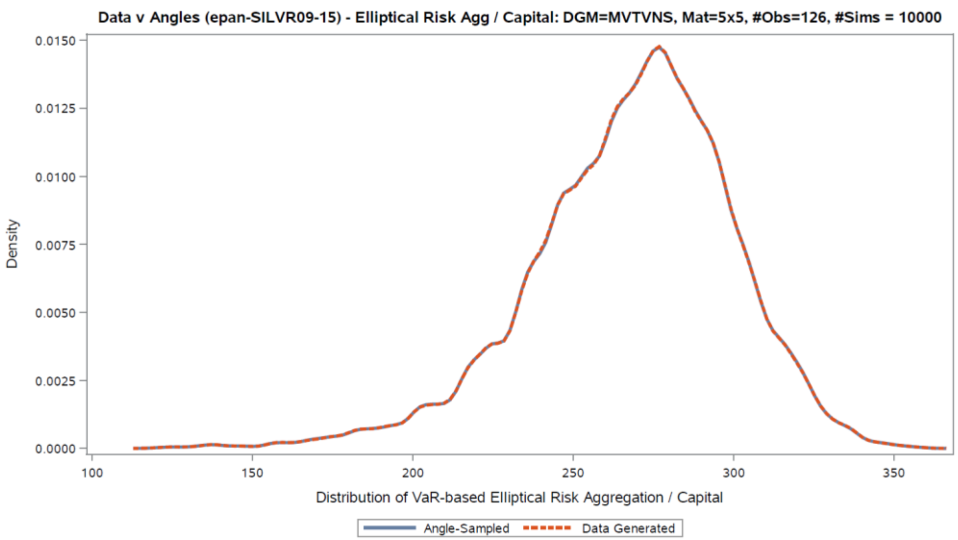
					lnMatPr	-2.251	-2.213	-7.009	-6.857	-13.451		-2.120	-2.345	-6.493	-7.395	-11.780
					FNorm	1.187	1.020	2.581	1.765	3.018		0.230	0.234	0.631	0.581	0.963
					Rnk_lnMat	2	1	4	3	5		1	2	3	4	5
					Rnk_FNorr	2	1	4	3	5		1	2	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.3391	-0.0058	0.6174	-0.2381	0.7001		0.29	0.19	0.37	0.11	0.41
20	-20	50	-50	60		0.3618	0.0062	0.6119	-0.2357	0.6939		0.26	0.18	0.33	0.12	0.36
20	-20	50	-50	60		0.3313	-0.0007	0.6003	-0.2301	0.6825		0.26	0.18	0.33	0.11	0.36
20	-20	50	-50	60		0.3350	-0.0048	0.5921	-0.2456	0.6747		0.26	0.18	0.33	0.12	0.36
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.4210	-0.0356	0.7581	-0.2325	0.8854		0.28	0.17	0.37	0.09	0.44
20	-20	50	-50	70		0.4189	-0.0311	0.7609	-0.2329	0.8859		0.28	0.17	0.37	0.09	0.44
20	-20	50	-50	70		0.4131	-0.0385	0.7476	-0.2218	0.8791		0.28	0.17	0.37	0.09	0.44
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.4969	-0.0757	0.8594	-0.1942	0.9754		0.28	0.16	0.38	0.08	0.48
20	-20	50	-50	80		0.4903	-0.0825	0.8642	-0.2021	0.9764		0.28	0.17	0.38	0.08	0.48
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.6249	-0.2052	0.9549	-0.1897	0.9996		0.29	0.16	0.40	0.06	0.55
						1	1	1	1	1		1	1	1	1	1

Elliptical Capital

n = 6 **n = 126**



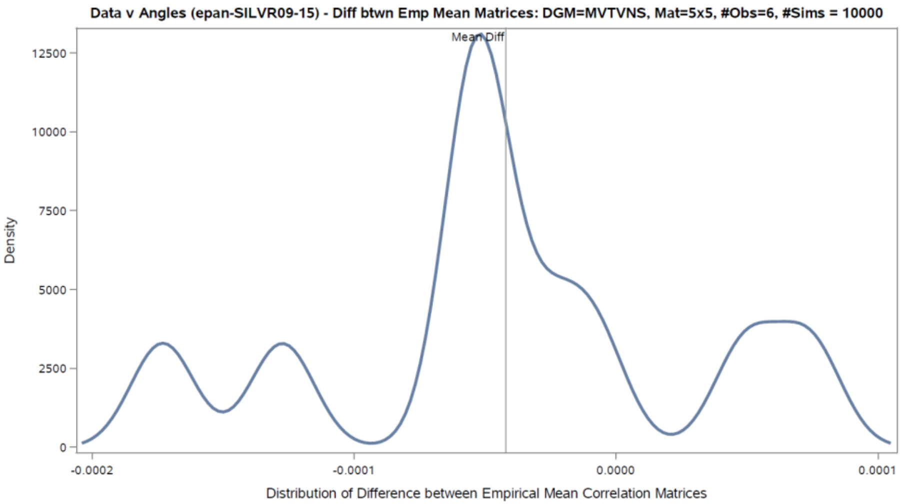
PRNG Seed = 12345
RunID = Fs5MMMblk



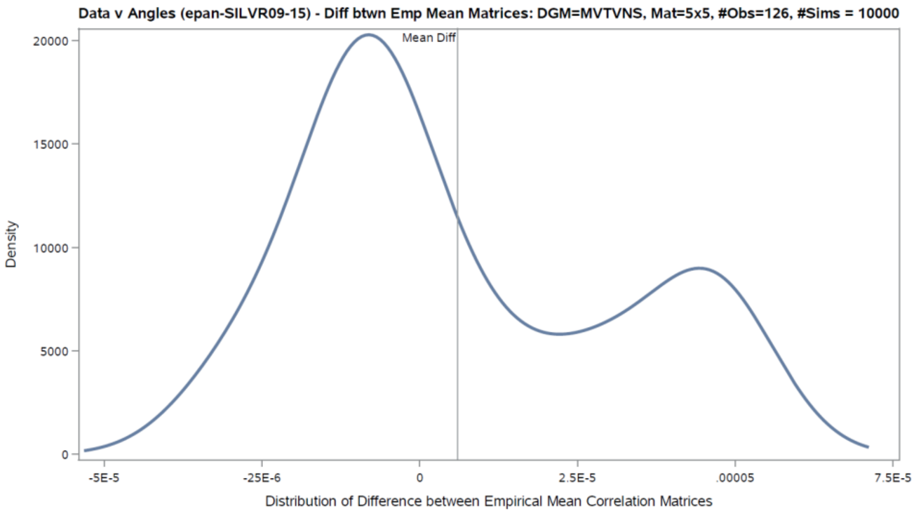
PRNG Seed = 12345
RunID = Fb5MMMblk

Difference Between Mean Empirical Matrices

n = 6 **n = 126**



PRNG Seed = 12345
RunID = Fs5MMMblk

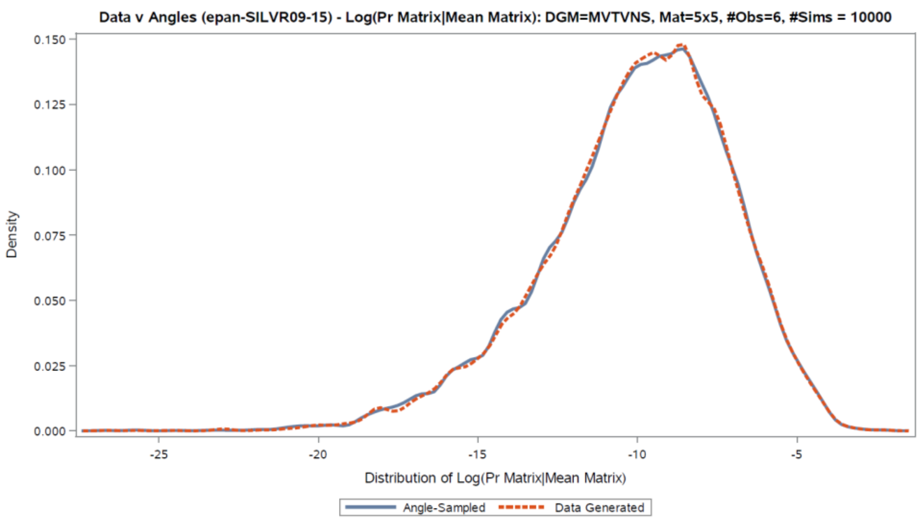


PRNG Seed = 12345
RunID = Fb5MMMblk

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

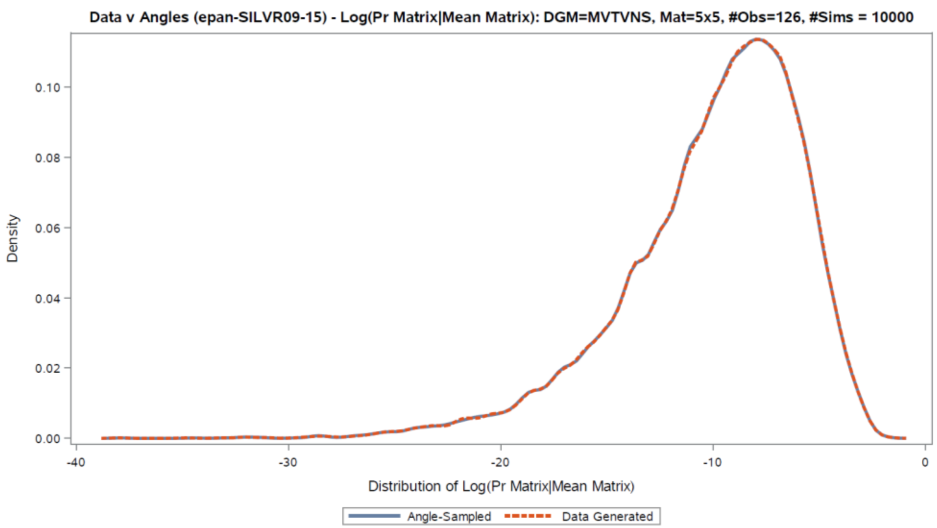
LNP

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

n = 126

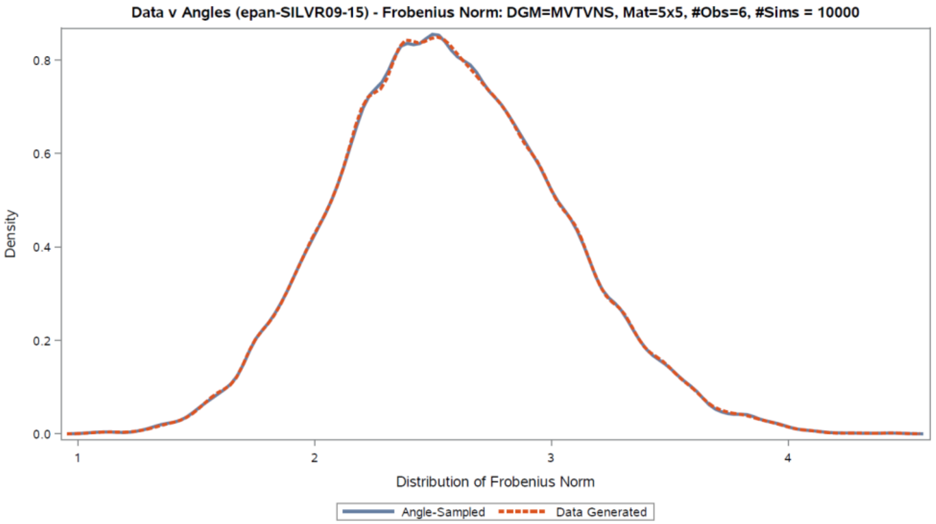


PRNG Seed = 12345
RunID = Fb5MMMblk

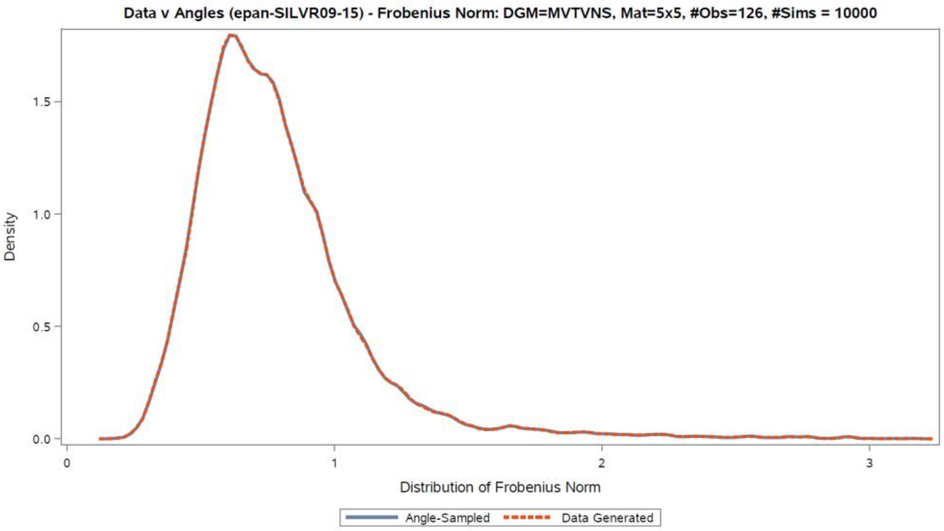
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

Euclidian/Frobenius Norm

n = 6 **n = 126**



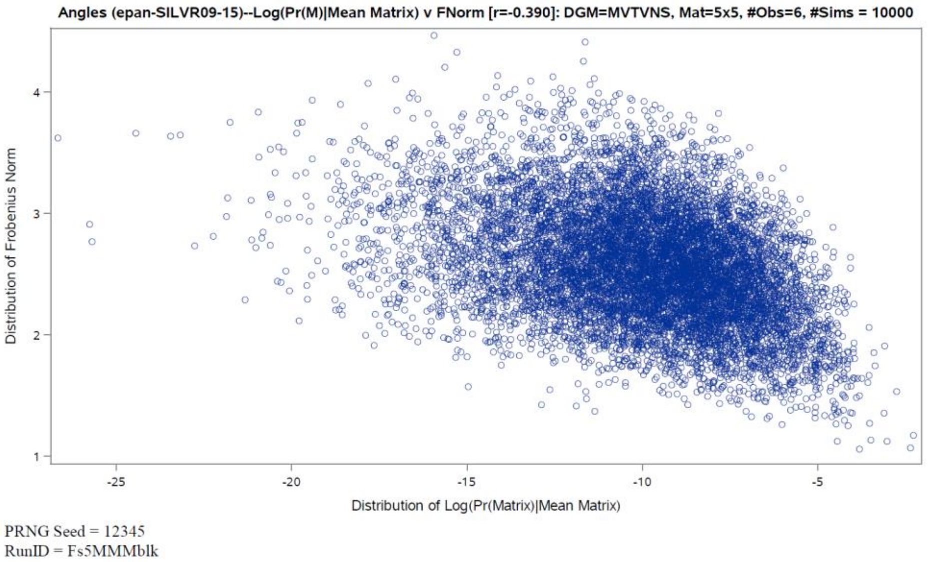
PRNG Seed = 12345
RunID = Fs5MMMblk



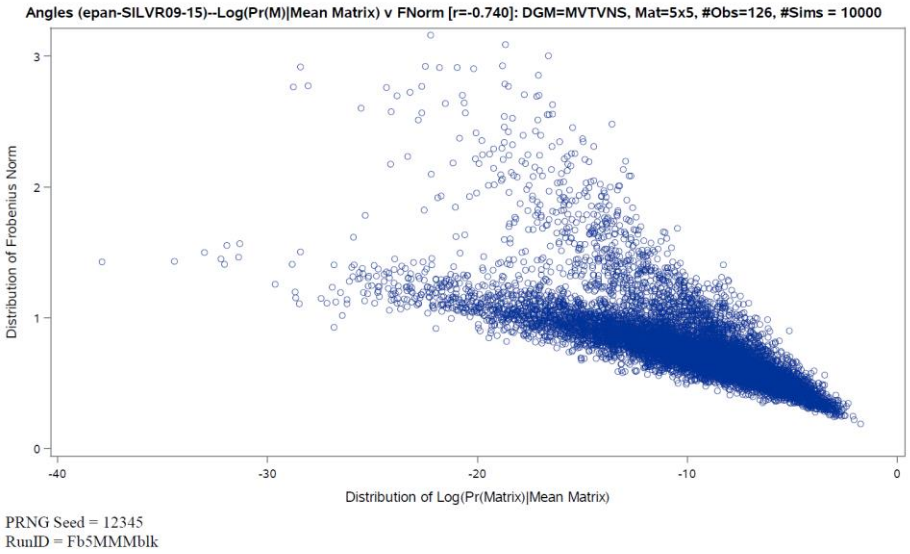
PRNG Seed = 12345
RunID = Fb5MMMblk

LNP v Euclidian/Frobenius Norm

n = 6

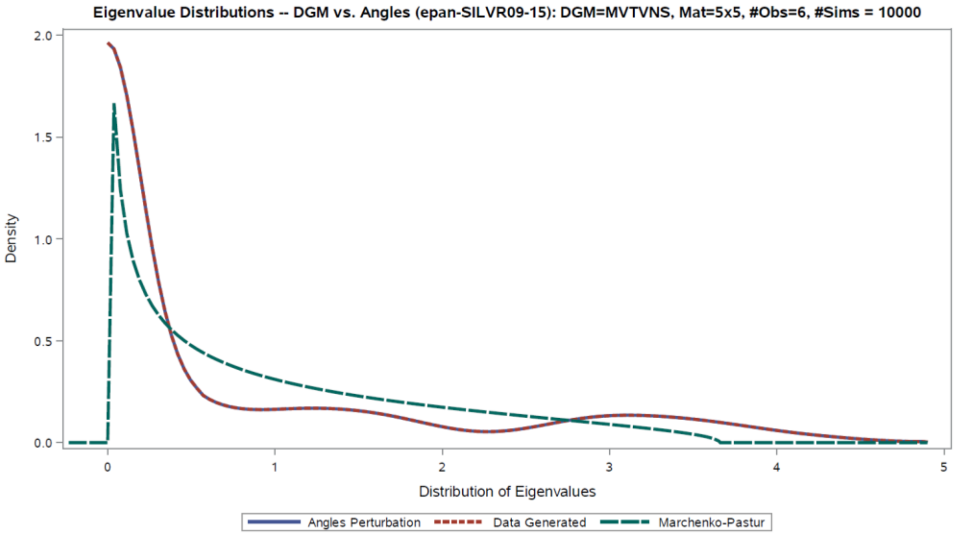


n = 126



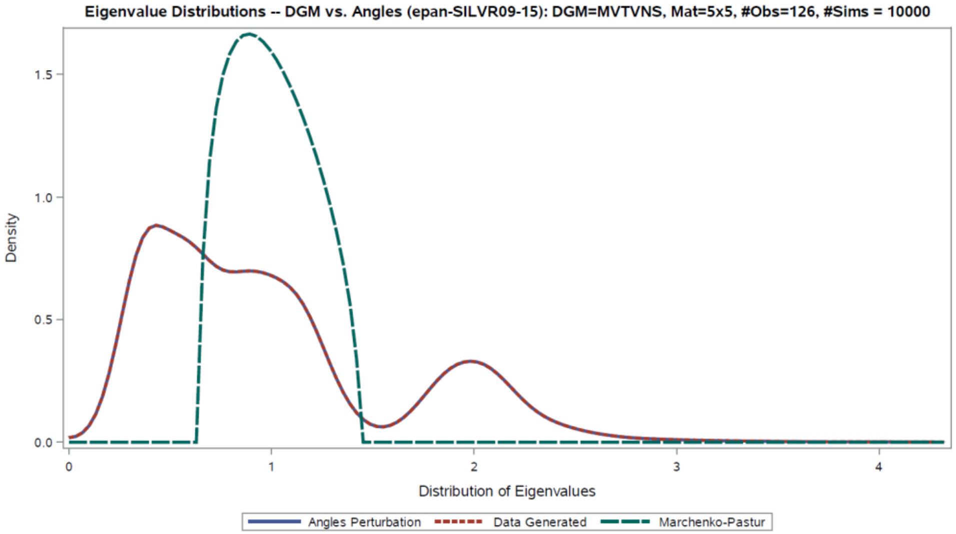
Spectral Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

n = 126

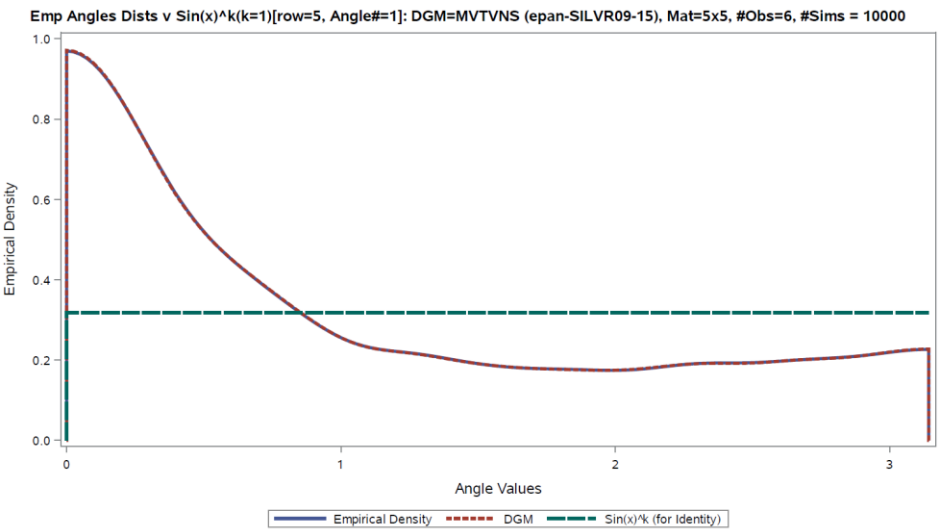


PRNG Seed = 12345
RunID = Fb5MMMblk

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

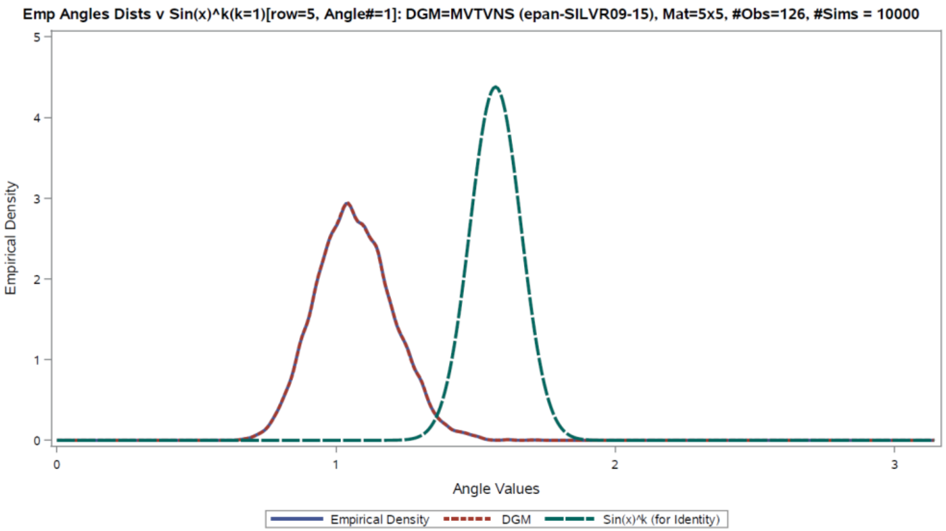
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

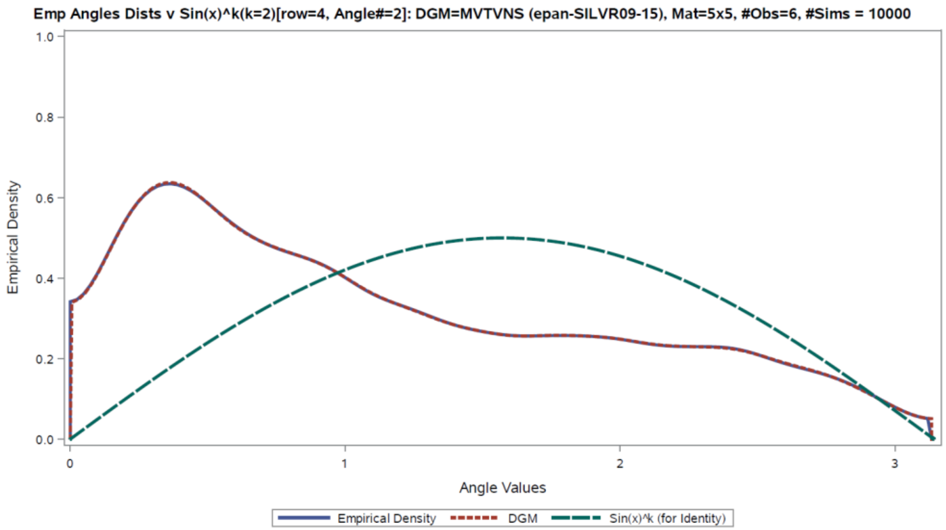
n = 126



PRNG Seed = 12345
RunID = Fb5MMMblk

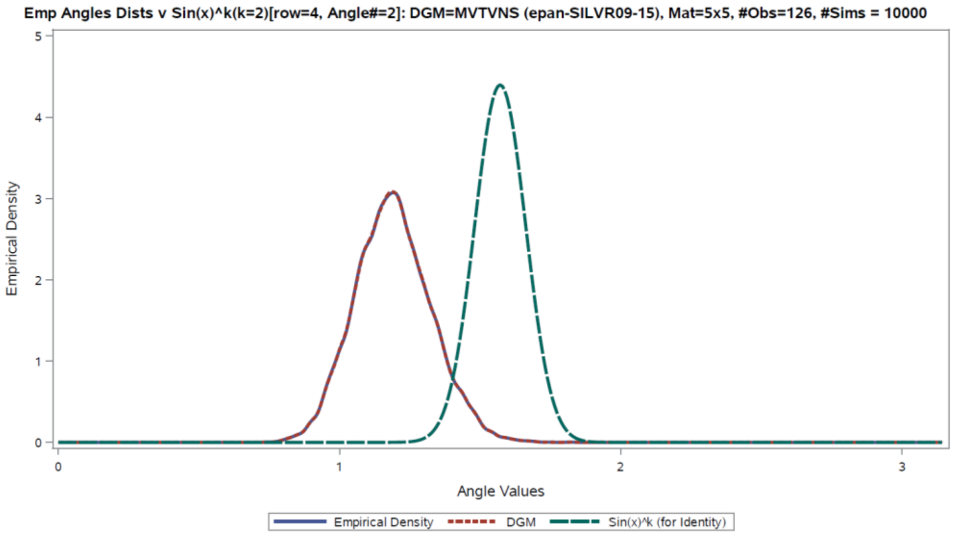
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

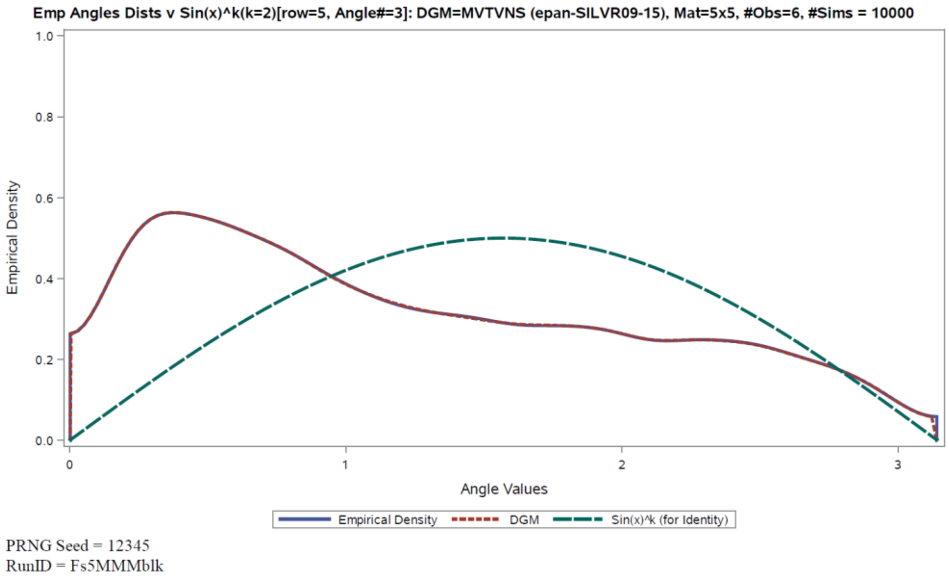
n = 126



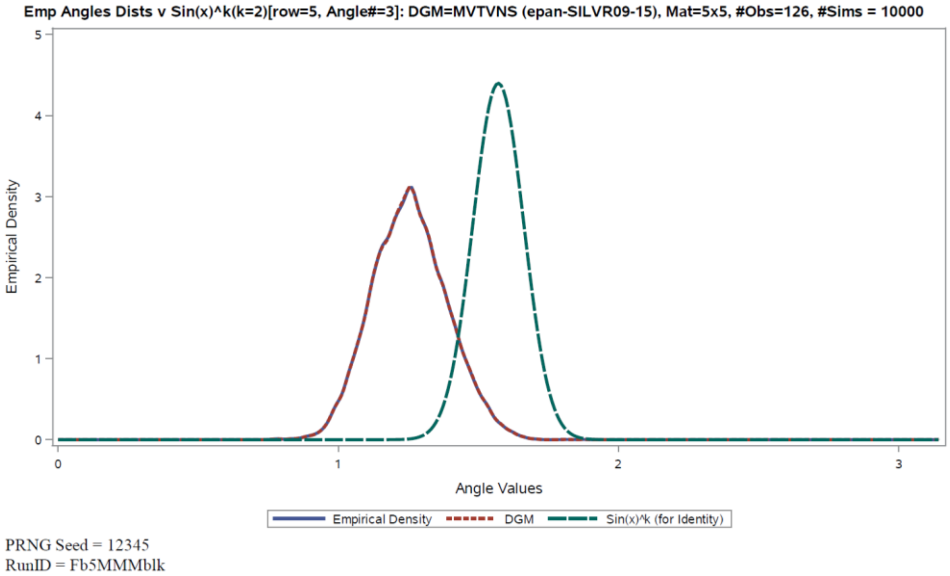
PRNG Seed = 12345
RunID = Fb5MMMblk

Angle Distributions

n = 6

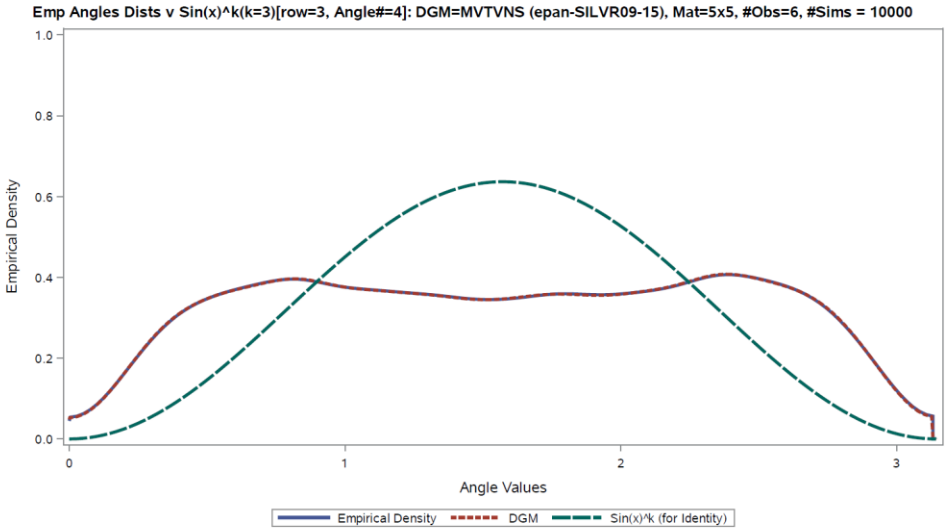


n = 126



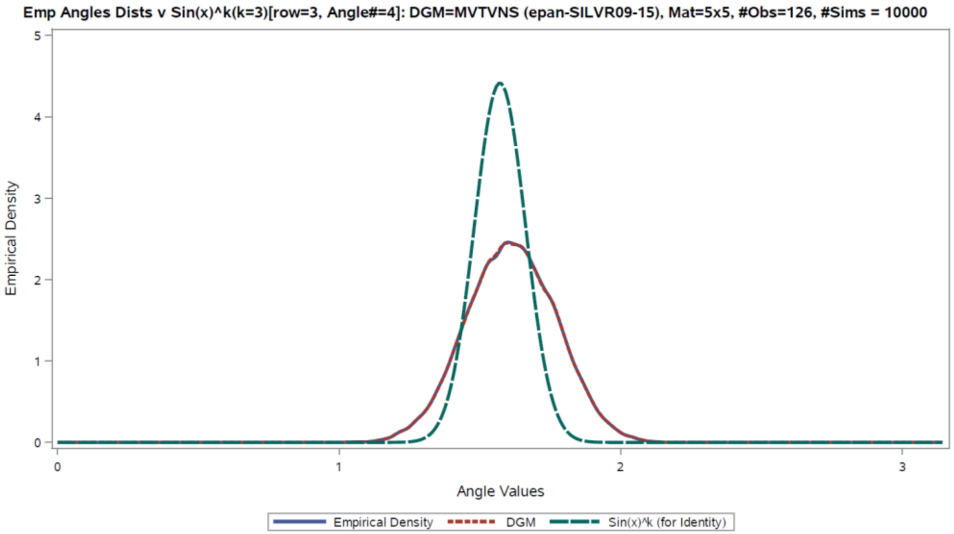
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

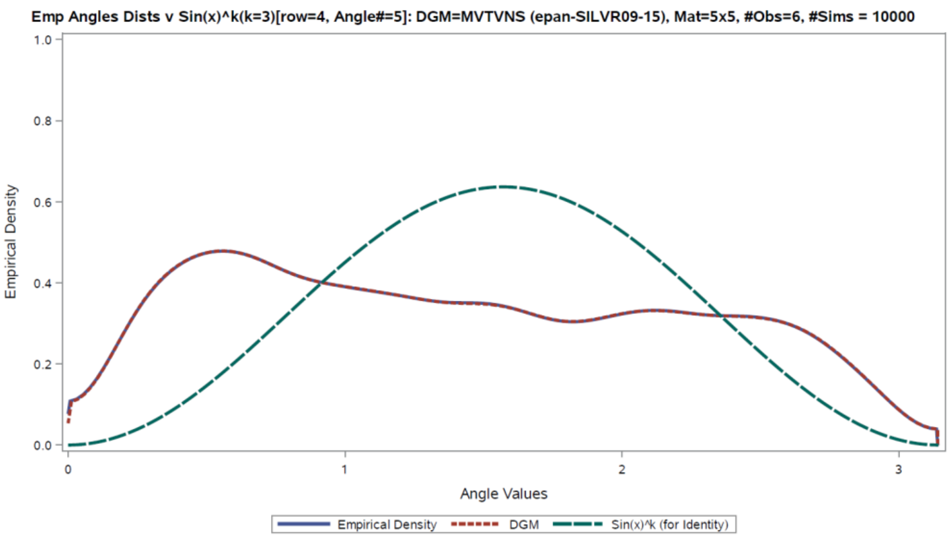
n = 126



PRNG Seed = 12345
RunID = Fb5MMMblk

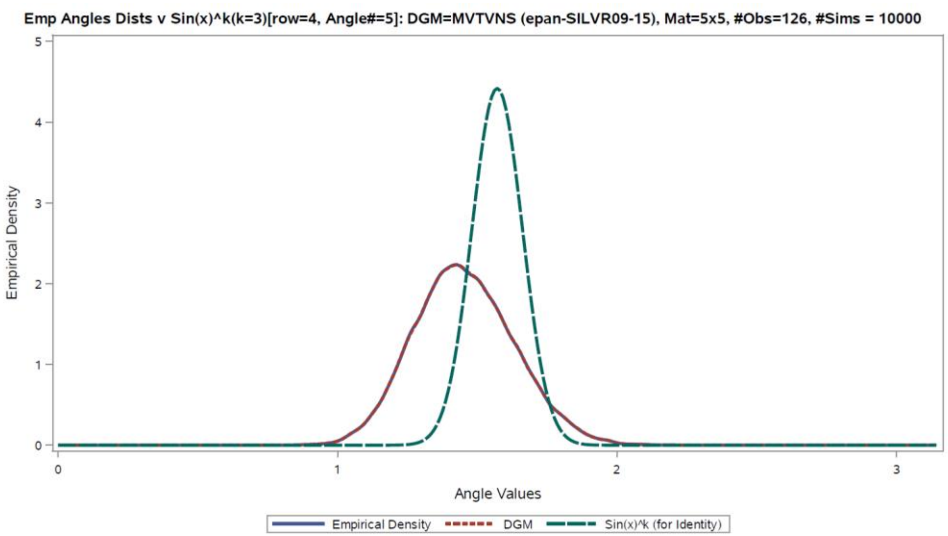
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

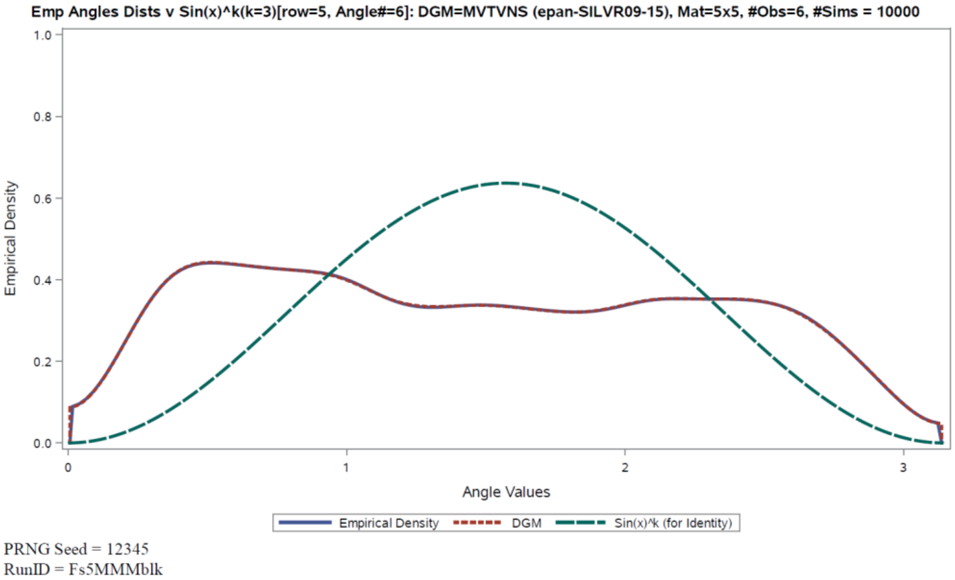
n = 126

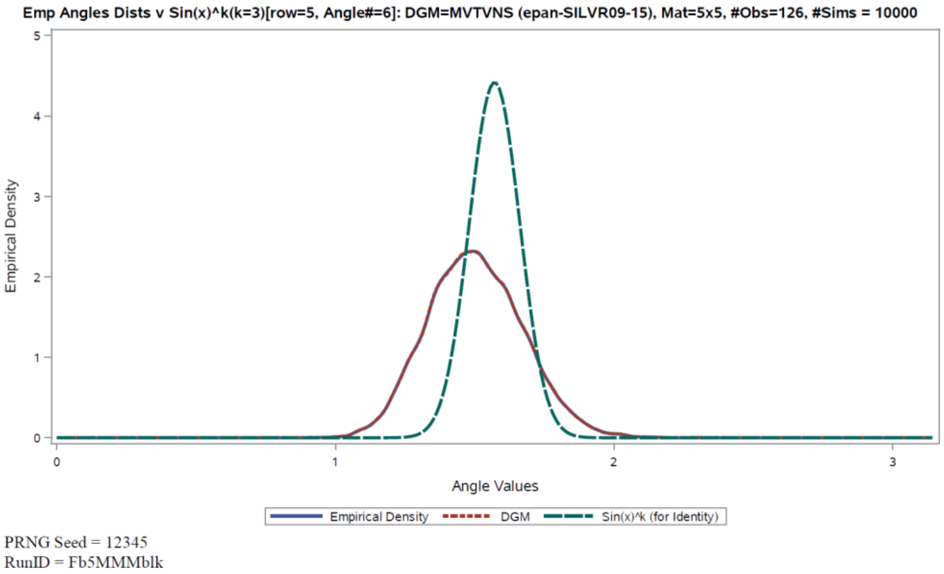


PRNG Seed = 12345
RunID = Fb5MMMblk

Angle Distributions

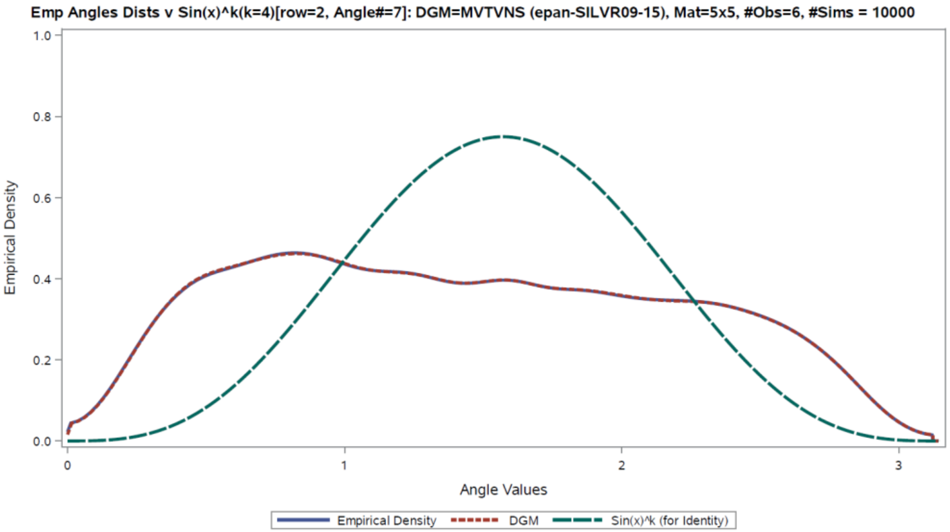
n = 6 **n = 126**





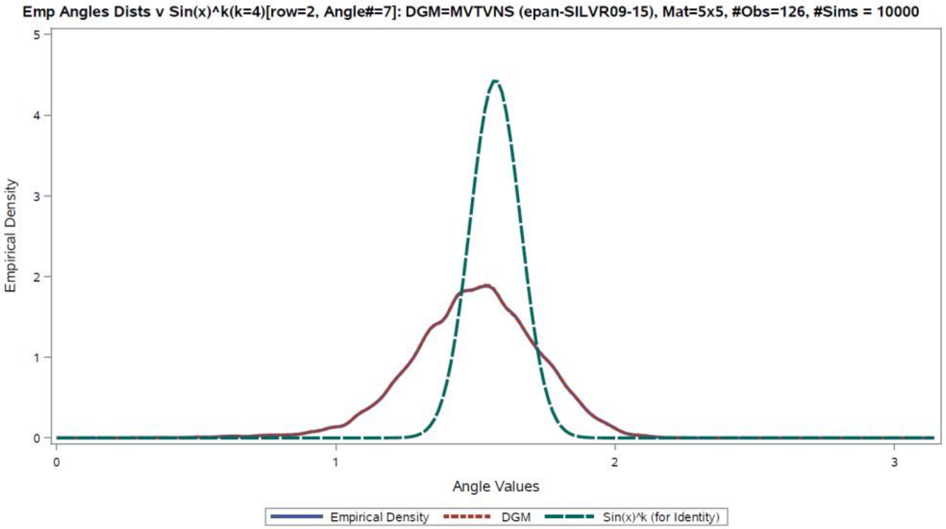
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

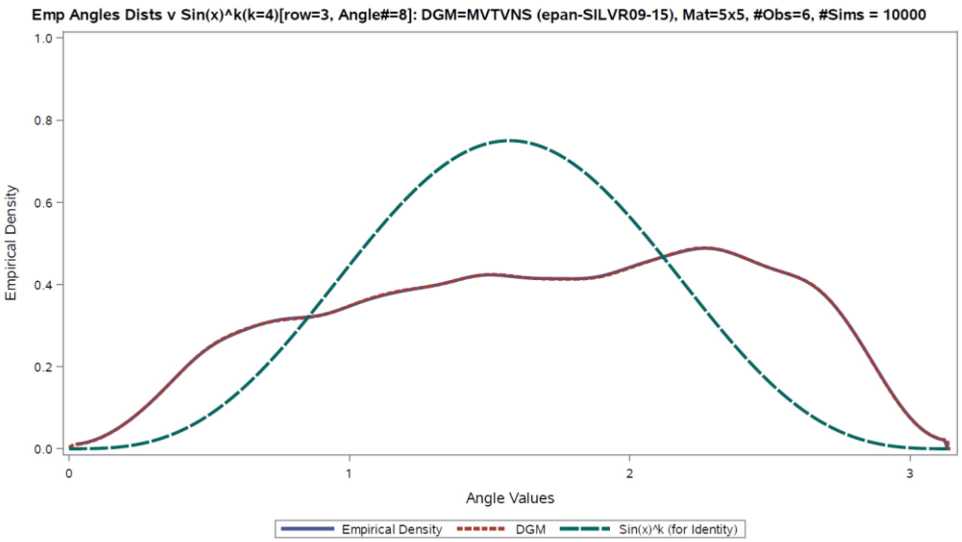
n = 126



PRNG Seed = 12345
RunID = Fb5MMMblk

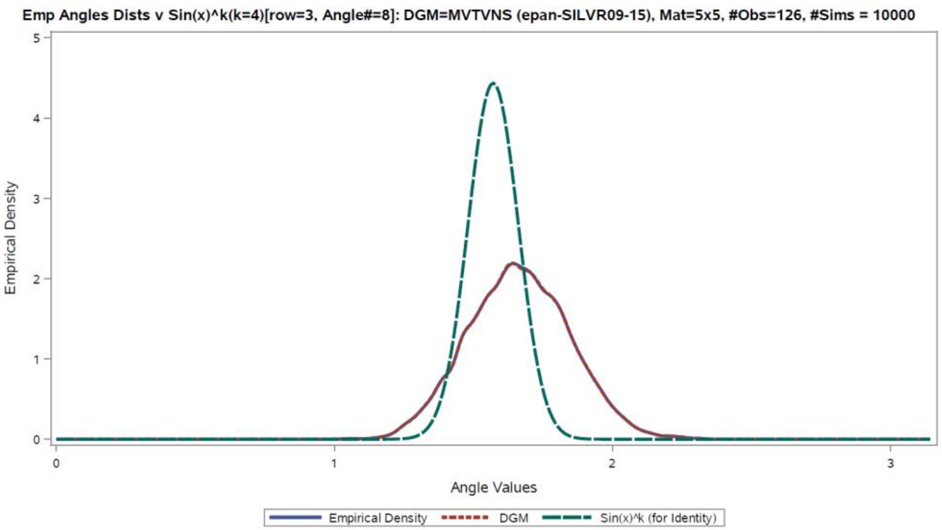
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Fs5MMMblk

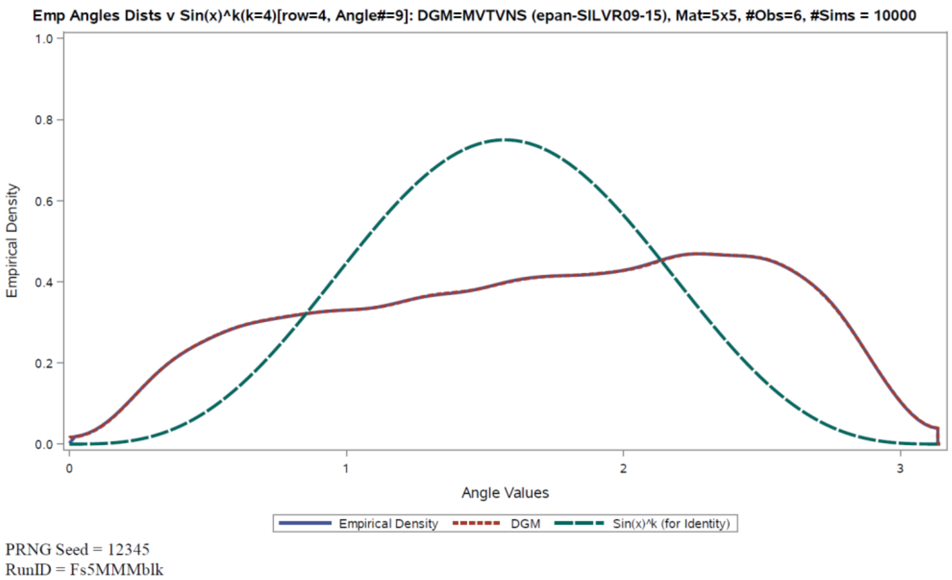
n = 126

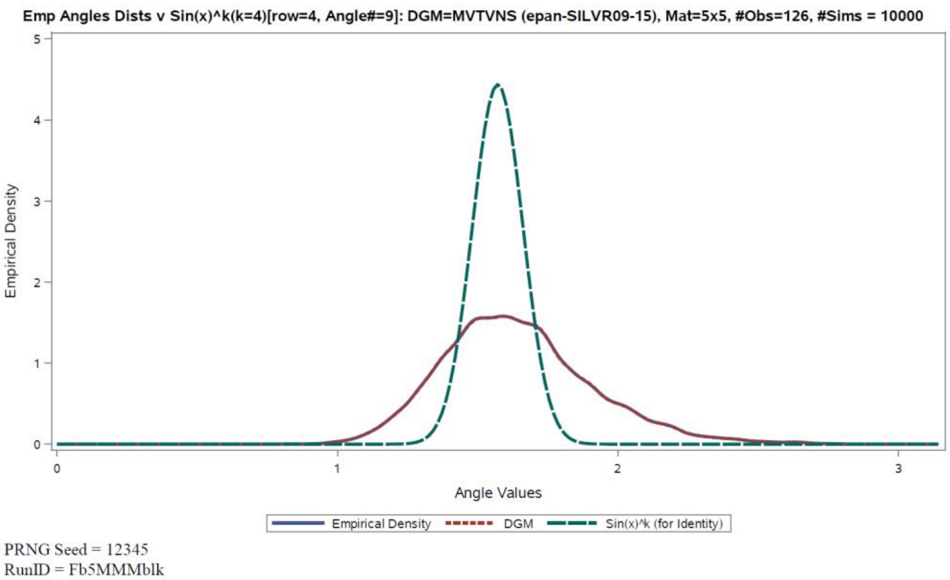


PRNG Seed = 12345
RunID = Fb5MMMblk

Angle Distributions

n = 6 **n = 126**

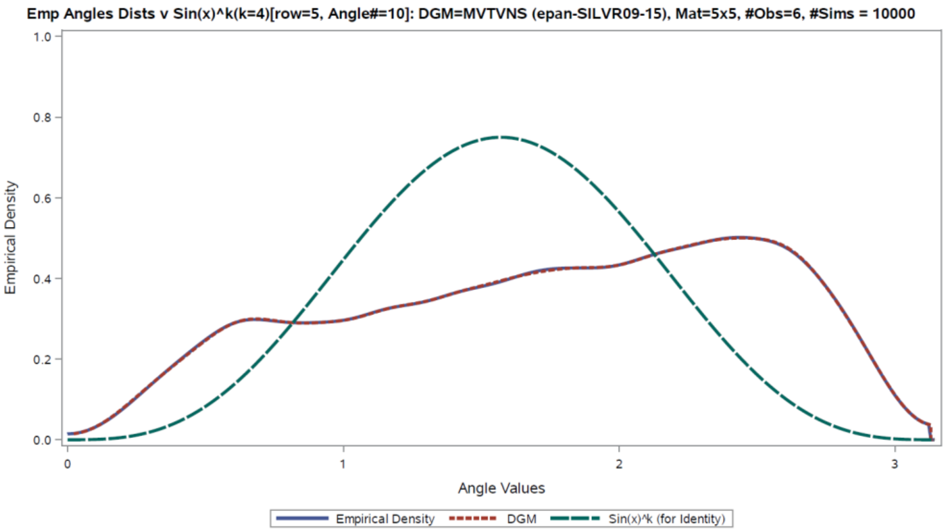




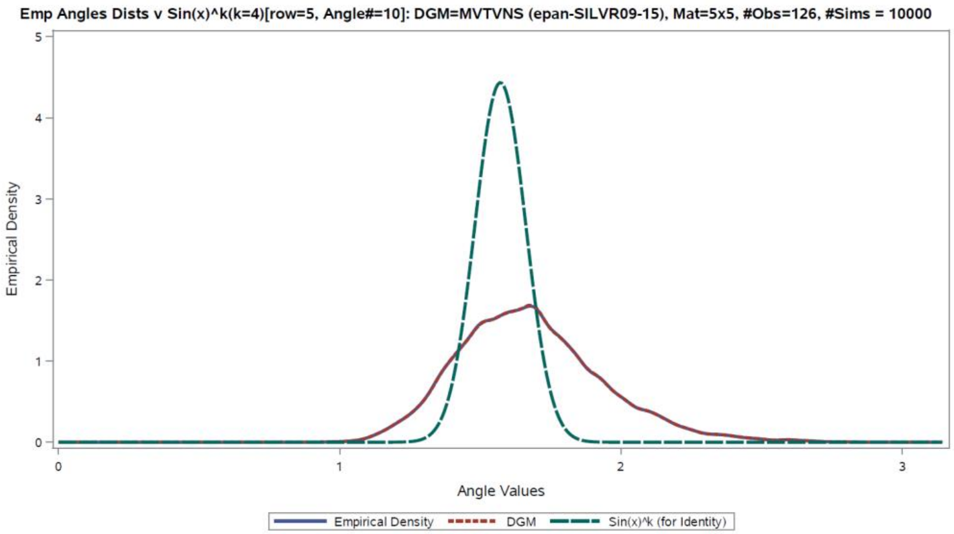
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

Angle Distributions

n = 6 **n = 126**



PRNG Seed = 12345
RunID = Fs5MMMblk



PRNG Seed = 12345
RunID = Fb5MMMblk

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

Correlation Matrices to CDF Matrices

Correlation Matrices | **n=6** | **n=126**

1	1	1	1	1												
0.1	0.2	0.3	0.4	0.3		0.53	0.49	0.45	0.40	0.45		0.4376	0.2705	0.1422	0.0672	0.1422
0.1	0.2	0.3	0.4	0.3		0.39	0.34	0.30	0.26	0.30		0.1493	0.0518	0.0116	0.0014	0.0116
0.1	0.2	0.3	0.4	0.3		0.38	0.35	0.31	0.27	0.31		0.2696	0.1437	0.0601	0.0177	0.0601
0.1	0.2	0.3	0.4	0.3		0.36	0.32	0.29	0.25	0.29		0.2125	0.099	0.0353	0.0077	0.0353
1	1	1	1	1												
0.1	0.2	0.3	0.4	0.4		0.46	0.44	0.41	0.39	0.37		0.206	0.0938	0.042	0.0184	0.0068
0.1	0.2	0.3	0.4	0.4		0.55	0.52	0.50	0.48	0.46		0.5734	0.4045	0.2671	0.169	0.0922
0.1	0.2	0.3	0.4	0.4		0.52	0.50	0.47	0.46	0.44		0.4607	0.2918	0.1656	0.0941	0.0426
1	1	1	1	1												
0.1	0.2	0.3	0.4	0.1		0.67	0.65	0.64	0.63	0.72		0.9824	0.953	0.9189	0.8742	0.9996
0.1	0.2	0.3	0.4	0.1		0.63	0.61	0.60	0.59	0.69		0.9535	0.8954	0.8262	0.7569	0.9996
1	1	1	1	1												
0.1	0.2	0.3	0.4	0.2		0.68	0.67	0.67	0.66	0.70		0.9978	0.9939	0.9902	0.986	0.9995
1	1	1	1	1												

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

CDF Matrices to Correlation Matrices

CDF Matrices | **n=6** | **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.18	0.40	-0.07	0.52	-0.80		0.07	0.12	0.01	0.15	-0.20
0.5	0.4	0.6	0.35	0.9		-0.17	0.07	-0.40	0.19	-0.87		-0.09	-0.05	-0.14	-0.02	-0.32
0.5	0.4	0.6	0.35	0.9		-0.18	0.06	-0.41	0.20	-0.88		-0.05	0.02	-0.11	0.05	-0.41
0.5	0.4	0.6	0.35	0.9		-0.25	-0.01	-0.47	0.12	-0.89		-0.09	-0.02	-0.14	0.01	-0.43
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		-0.04	0.27	-0.24	-0.15	0.48		-0.05	-0.01	-0.08	-0.08	-0.10
0.5	0.4	0.6	0.6	0.8		0.19	0.47	-0.01	0.06	0.53		0.12	0.17	0.08	0.08	0.06
0.5	0.4	0.6	0.6	0.8		0.11	0.40	-0.09	-0.05	0.53		0.07	0.11	0.03	0.03	0.02
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.56	0.75	0.43	0.72	0.98		0.37	0.40	0.35	0.39	0.51
0.5	0.4	0.6	0.4	0.3		0.49	0.70	0.37	0.69	0.98		0.31	0.34	0.29	0.34	0.47
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.81	0.94	0.58	0.81	0.93		0.55	0.59	0.52	0.55	0.50
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case F

CDF %Shift Matrices to Correlation Matrices

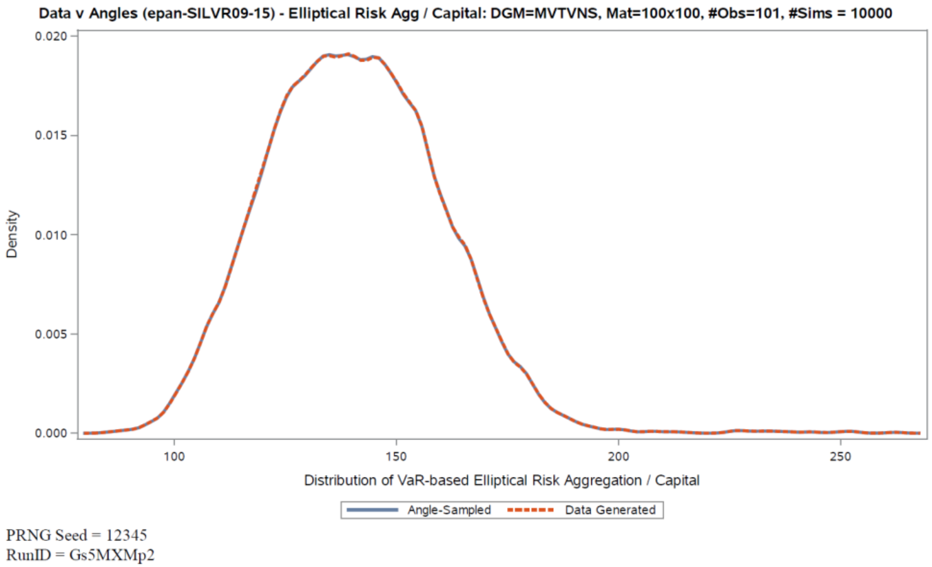
CDF %Shift Matrices | **n=6** | **n=126**

					lnMatPr	-2.369	-2.101	-7.529	-6.449	-14.780		-2.292	-2.175	-7.204	-6.731	-12.865
					FNorm	1.297	0.935	2.820	1.632	3.266		0.218	0.190	0.593	0.500	0.847
					Rnk_lnMat	2	1	4	3	5		2	1	4	3	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.364	-0.129	0.670	-0.494	0.749		0.13	0.02	0.22	-0.07	0.25
20	-20	50	-50	60		0.118	-0.355	0.459	-0.644	0.572		-0.05	-0.14	0.03	-0.22	0.06
20	-20	50	-50	60		0.129	-0.357	0.502	-0.656	0.616		0.00	-0.13	0.10	-0.24	0.14
20	-20	50	-50	60		0.081	-0.390	0.458	-0.685	0.594		-0.04	-0.16	0.07	-0.27	0.10
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.283	-0.234	0.723	-0.119	0.864		-0.01	-0.09	0.07	-0.14	0.13
20	-20	50	-50	70		0.396	-0.125	0.804	-0.052	0.913		0.14	0.05	0.24	-0.01	0.31
20	-20	50	-50	70		0.349	-0.177	0.766	-0.048	0.900		0.09	0.00	0.18	-0.04	0.25
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.655	0.236	0.934	0.498	0.987		0.40	0.35	0.46	0.33	0.54
20	-20	50	-50	80		0.617	0.215	0.922	0.522	0.988		0.35	0.30	0.41	0.28	0.48
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.880	0.307	0.992	0.551	0.999		0.60	0.53	0.67	0.49	0.77
						1	1	1	1	1		1	1	1	1	1

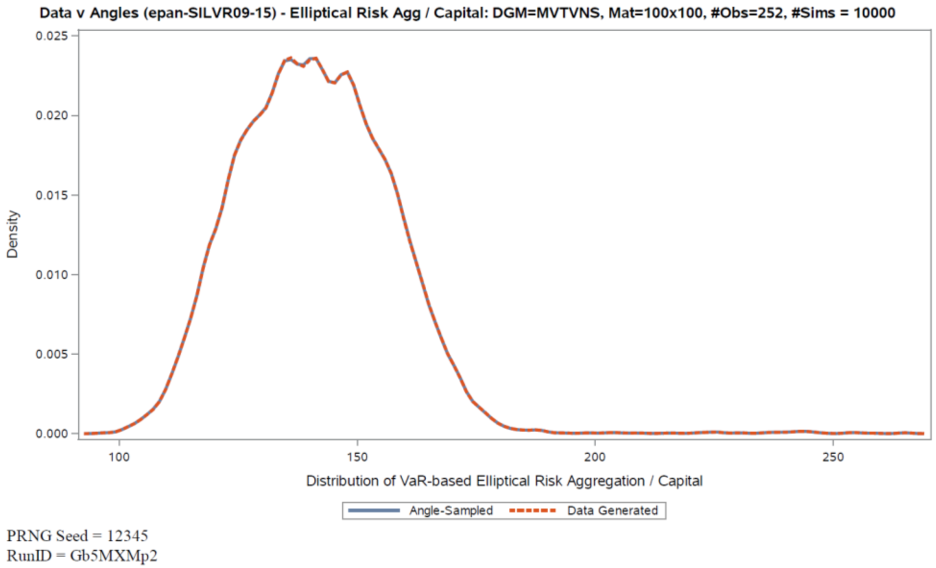
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Elliptical Capital

n = 101

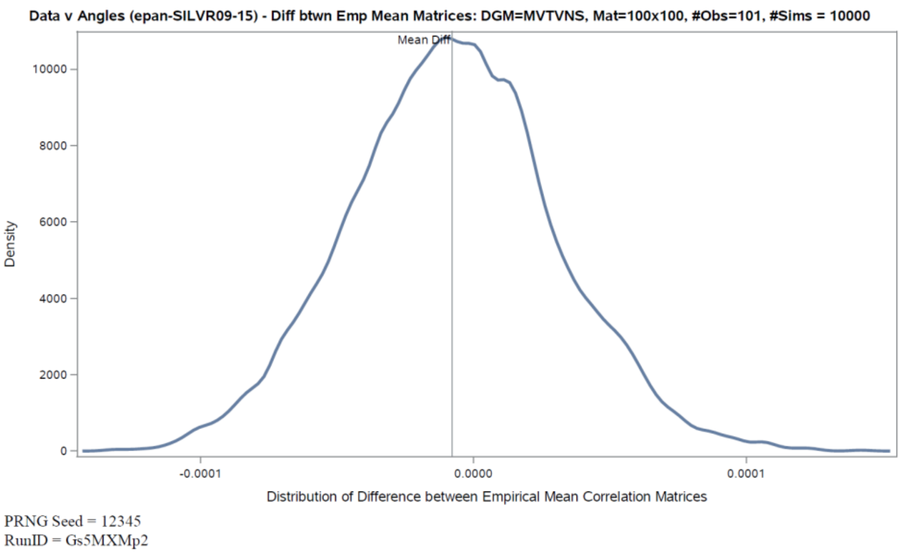


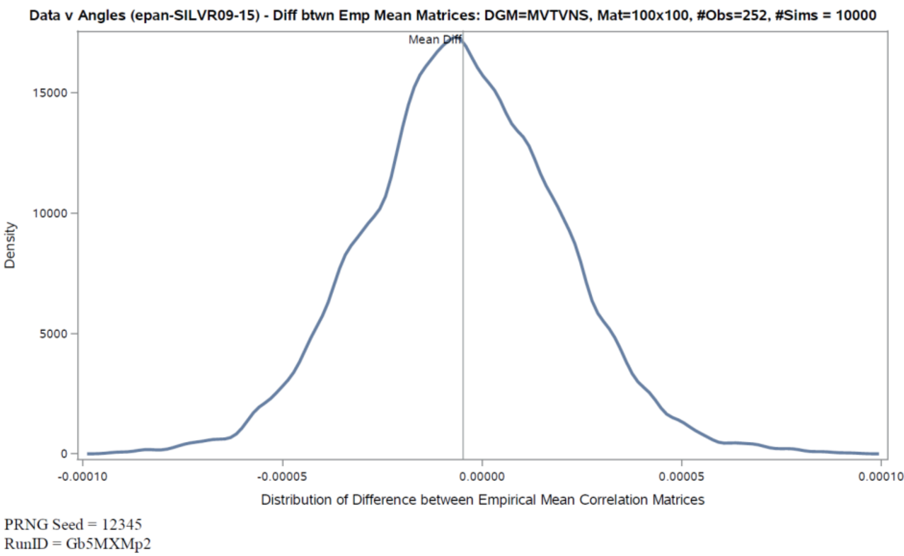
n = 252



Difference Between Mean Empirical Matrices

n = 101 | **n = 252**

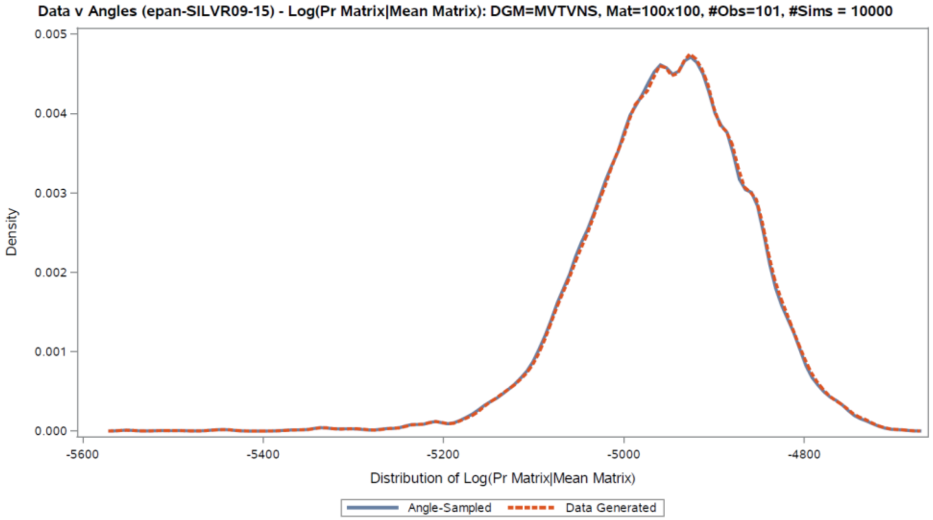




XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

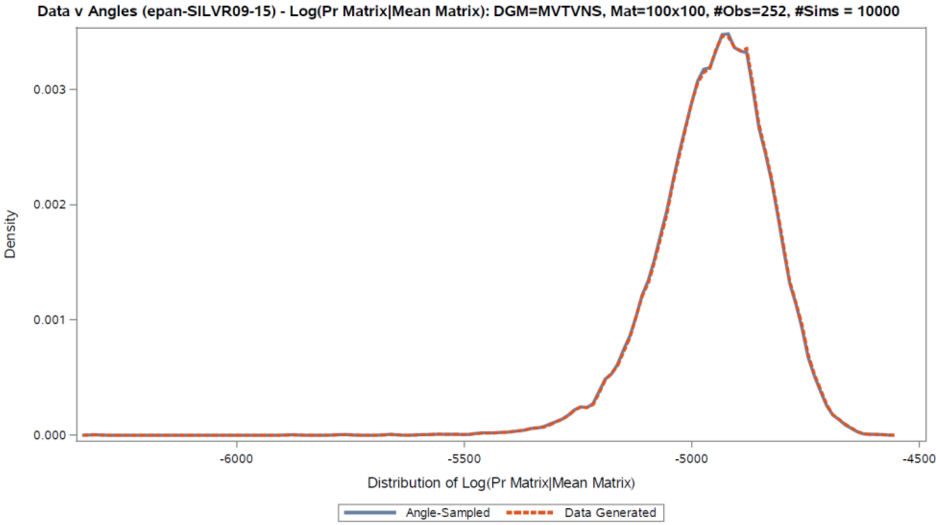
LNP

n = 101



PRNG Seed = 12345
RunID = Gs5MXMp2

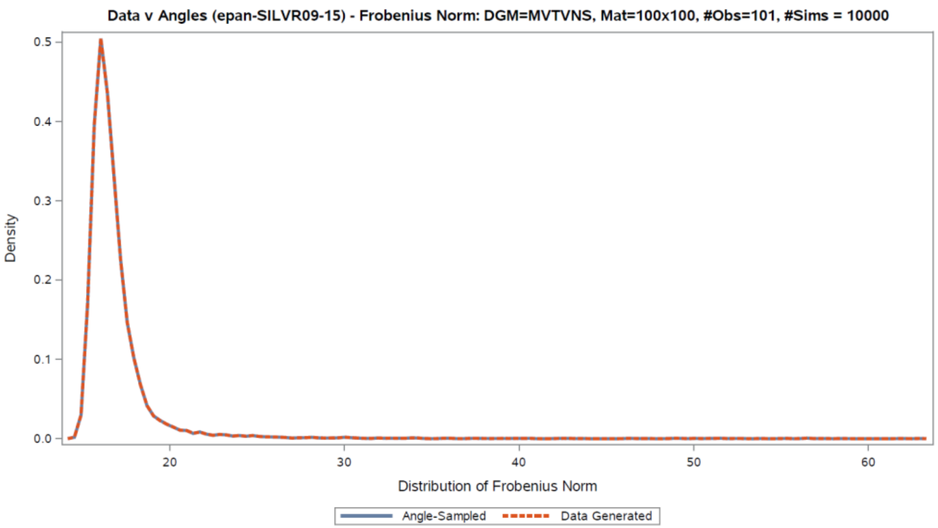
n = 252



PRNG Seed = 12345
RunID = Gb5MXMp2

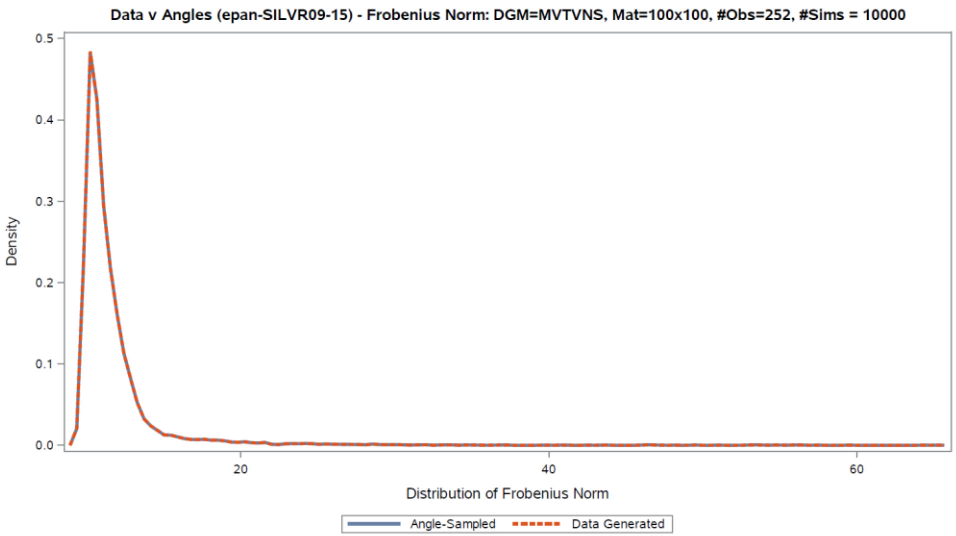
Euclidian/Frobenius Norm

n = 101



PRNG Seed = 12345
RunID = Gs5MXMp2

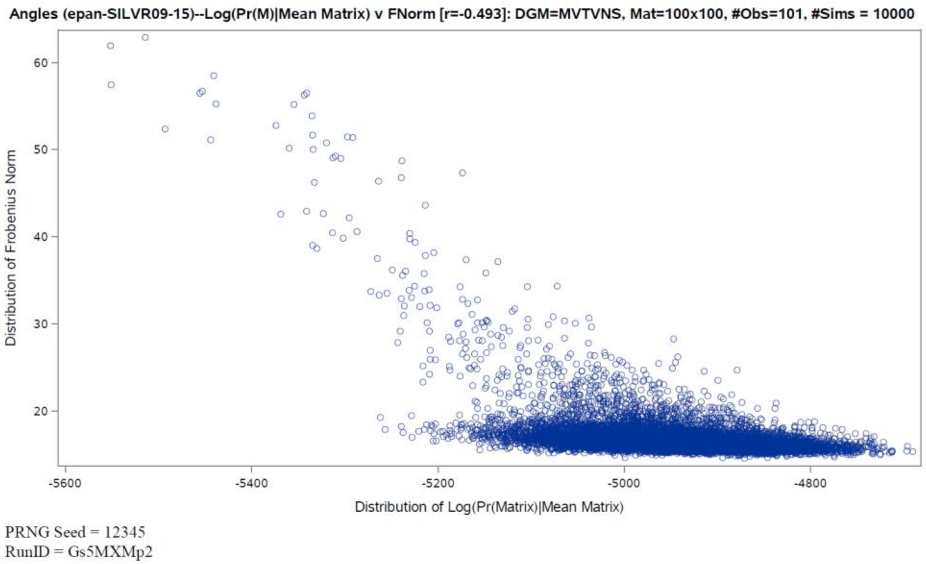
n = 252



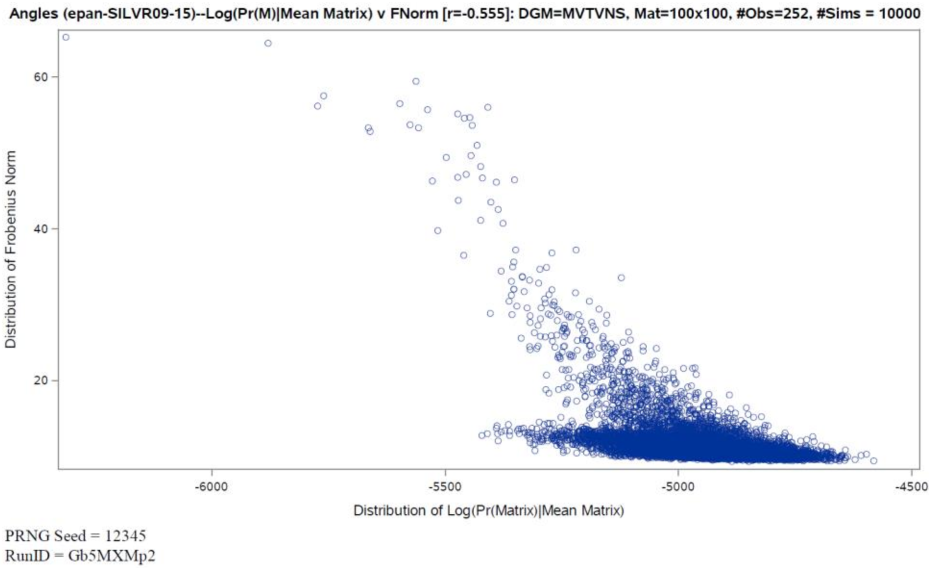
PRNG Seed = 12345
RunID = Gb5MXMp2

LNP v Euclidian/Frobenius Norm

n = 101

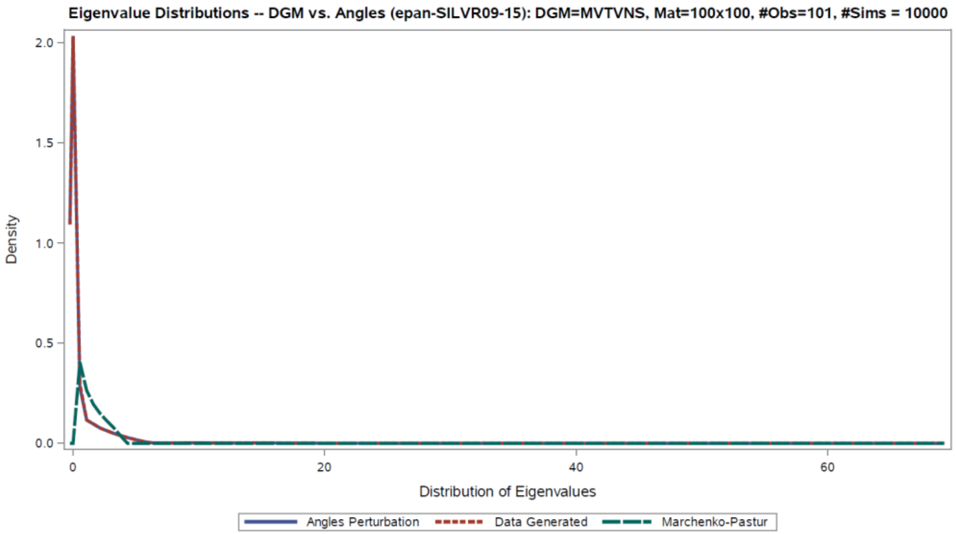


n = 252

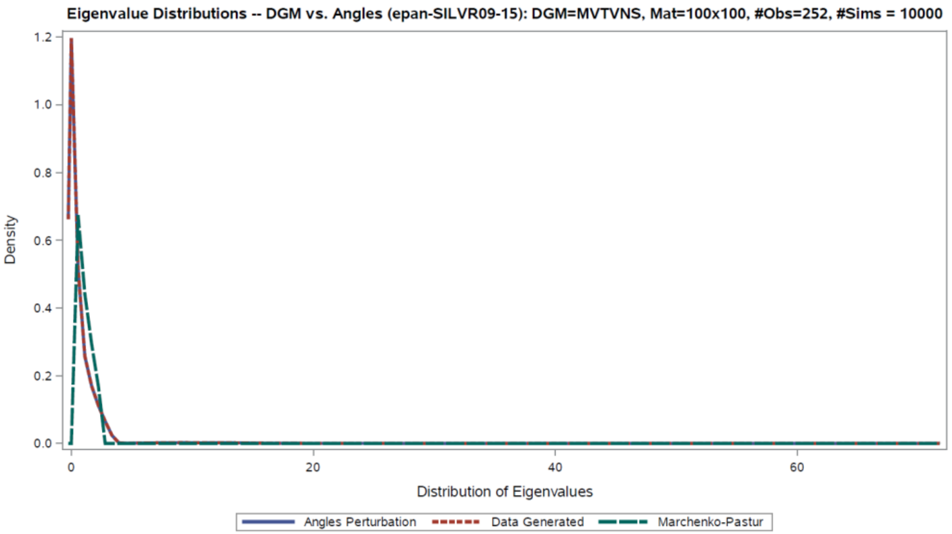


Spectral Distributions

n = 101 **n = 252**



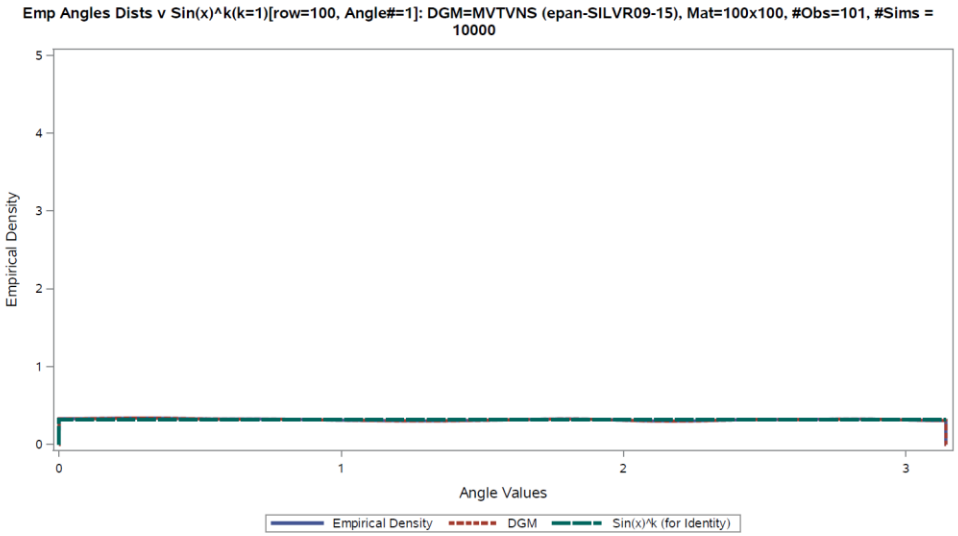
PRNG Seed = 12345
RunID = Gs5MXMp2



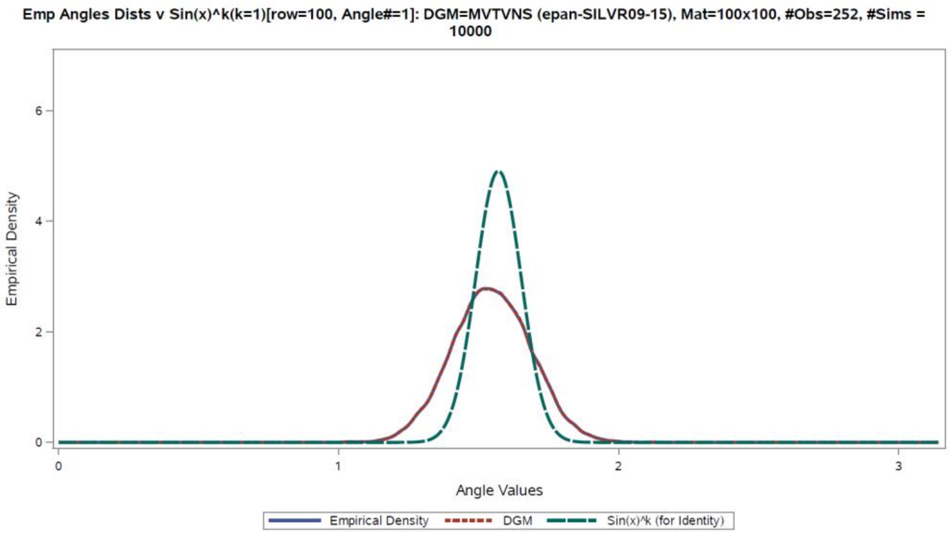
PRNG Seed = 12345
RunID = Gb5MXMp2

Angle Distributions

n = 101 **n = 252**



PRNG Seed = 12345
RunID = Gs5MXMp2

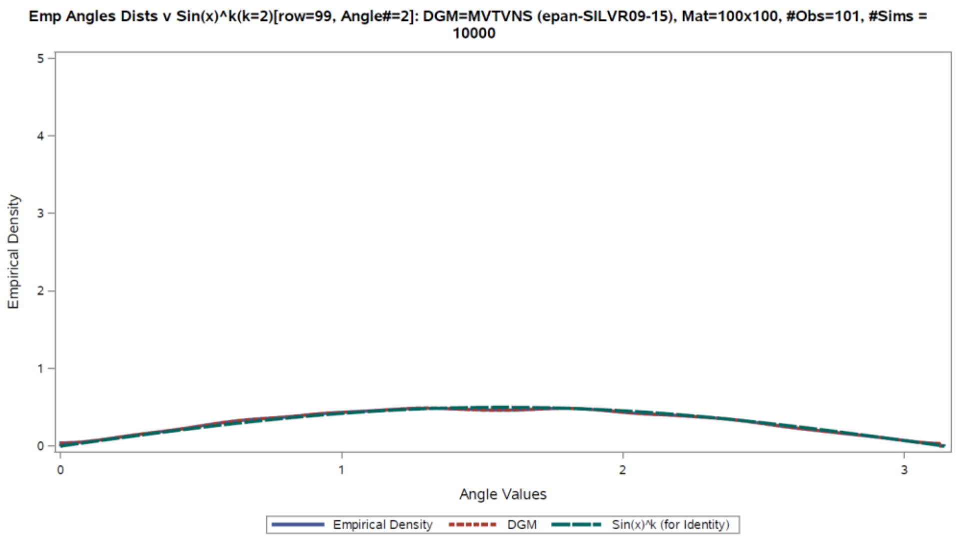


PRNG Seed = 12345
RunID = Gb5MXMp2

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

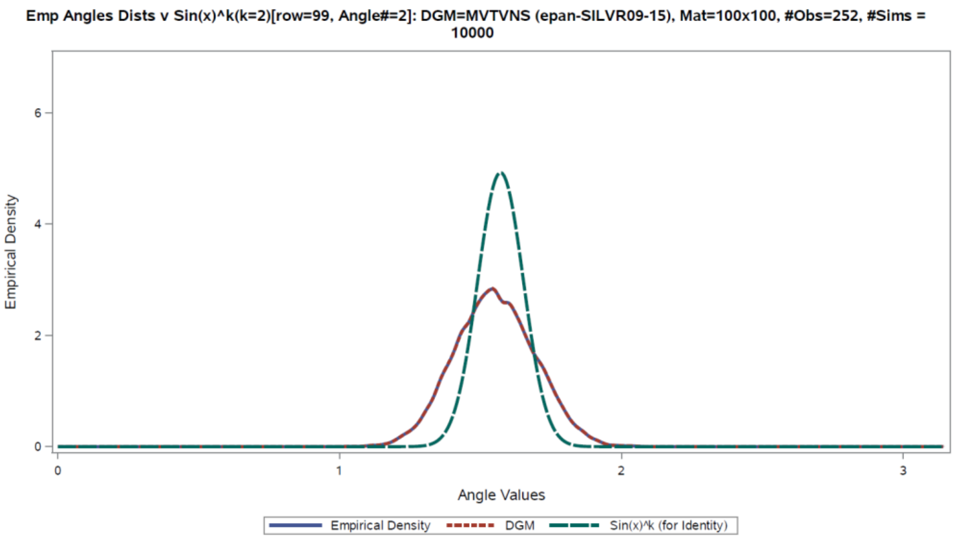
Angle Distributions

n = 101



PRNG Seed = 12345
RunID = Gs5MXMp2

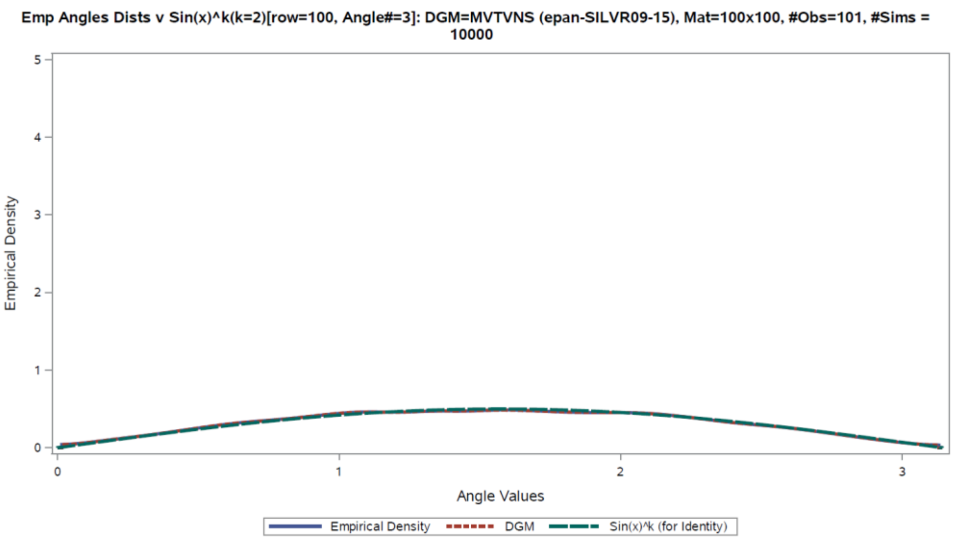
n = 252



PRNG Seed = 12345
RunID = Gb5MXMp2

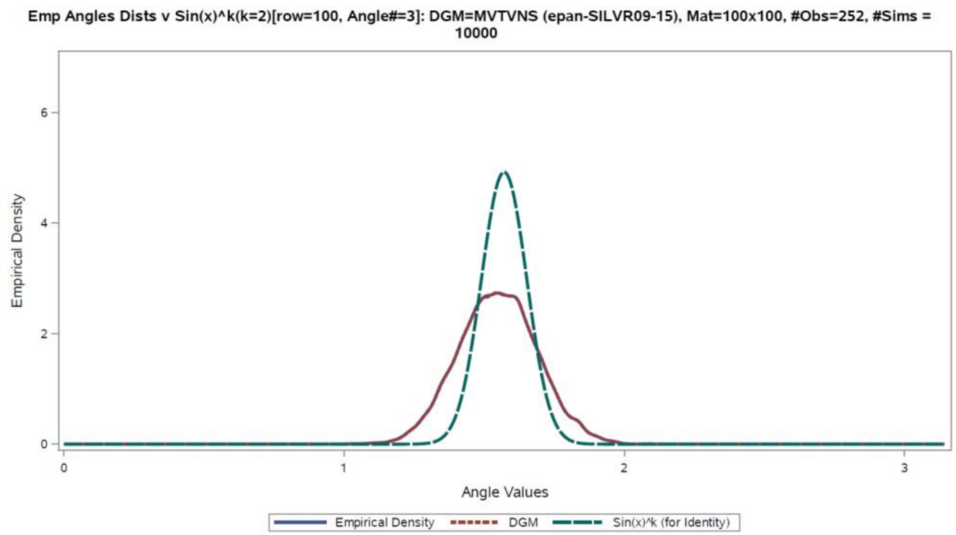
Angle Distributions

n = 101



PRNG Seed = 12345
RunID = Gs5MXMp2

n = 252

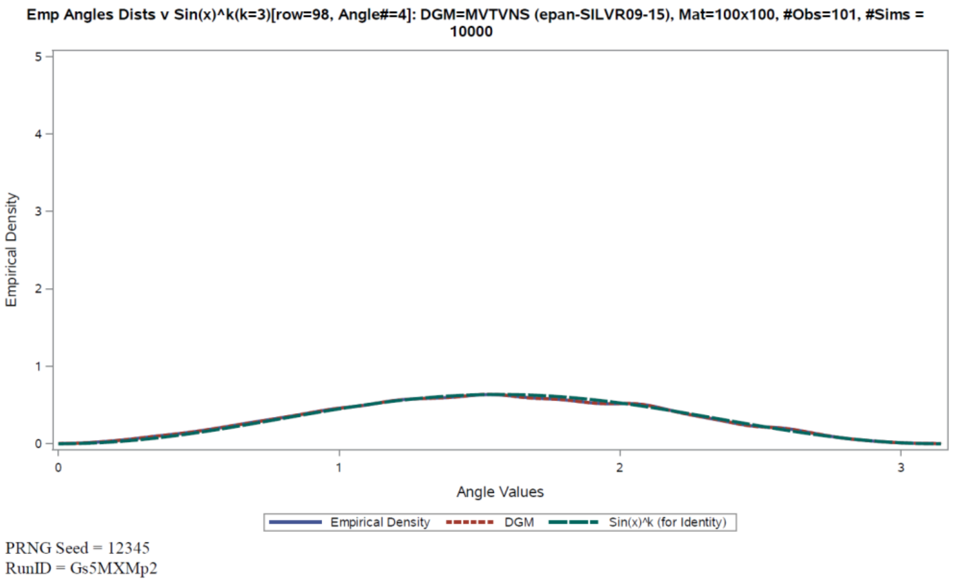


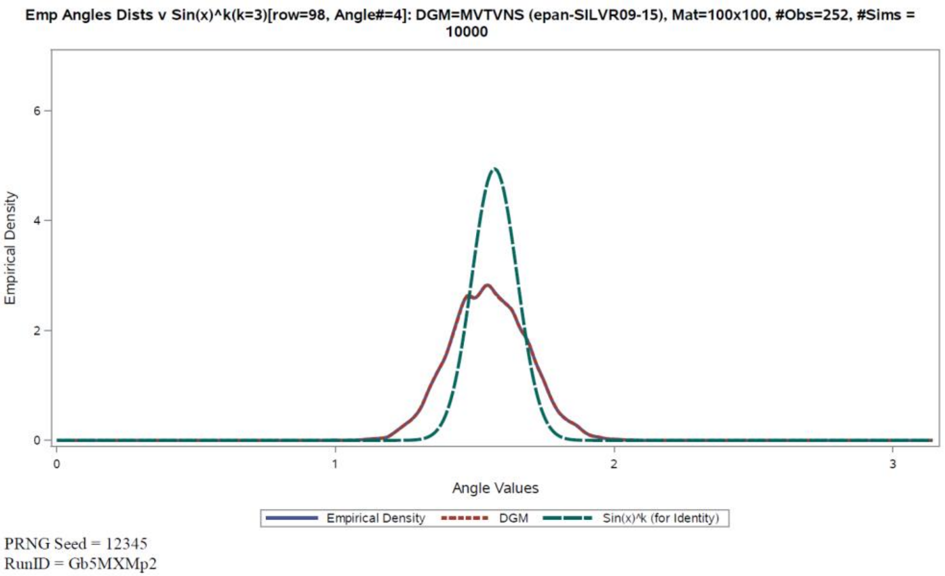
PRNG Seed = 12345
RunID = Gb5MXMp2

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Angle Distributions

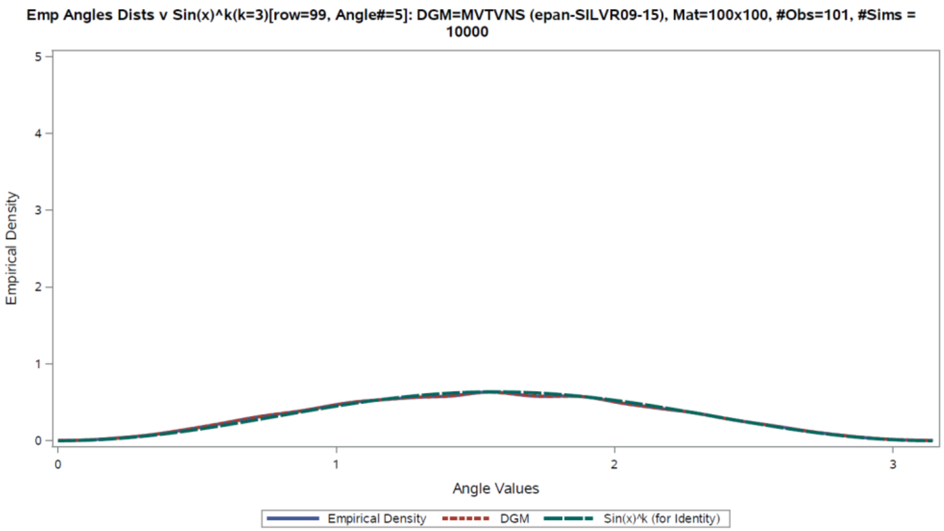
n = 101 **n = 252**



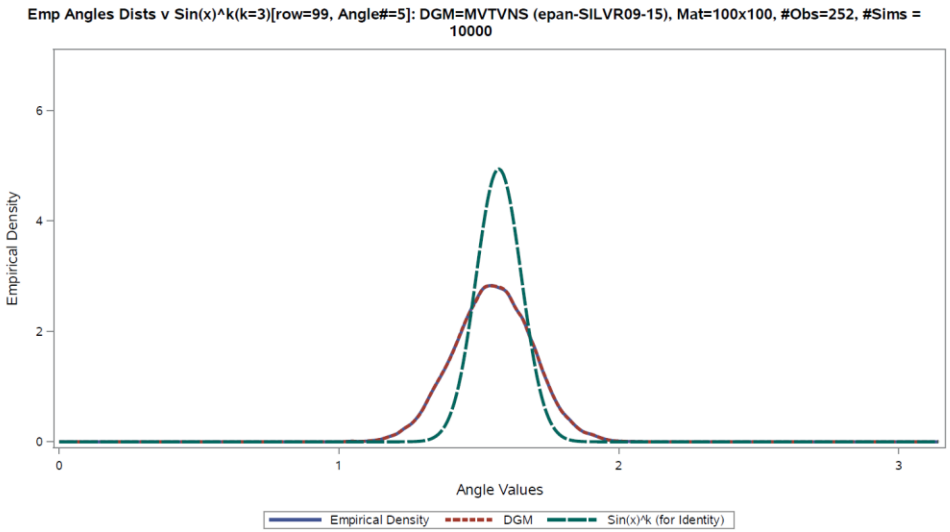


Angle Distributions

n = 101 | **n = 252**



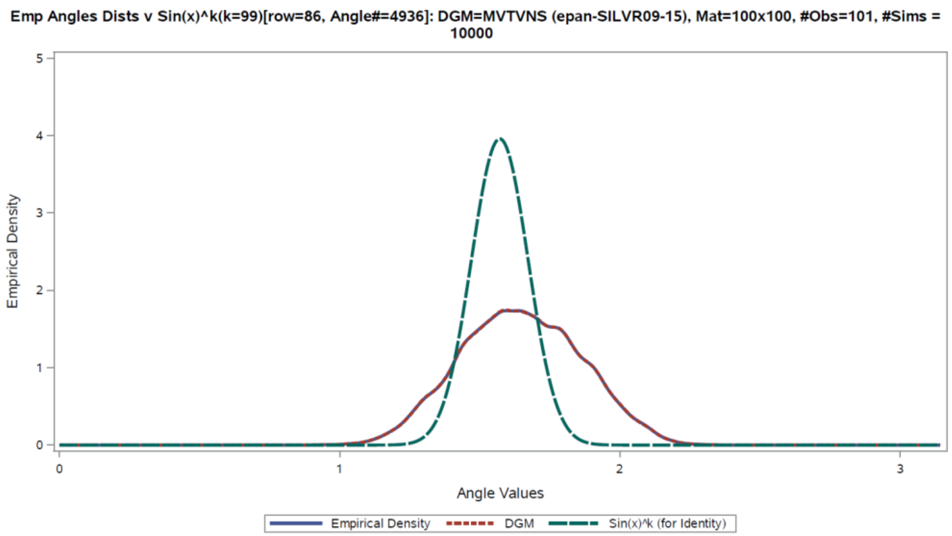
PRNG Seed = 12345
RunID = Gs5MXMp2



PRNG Seed = 12345
RunID = Gb5MXMp2

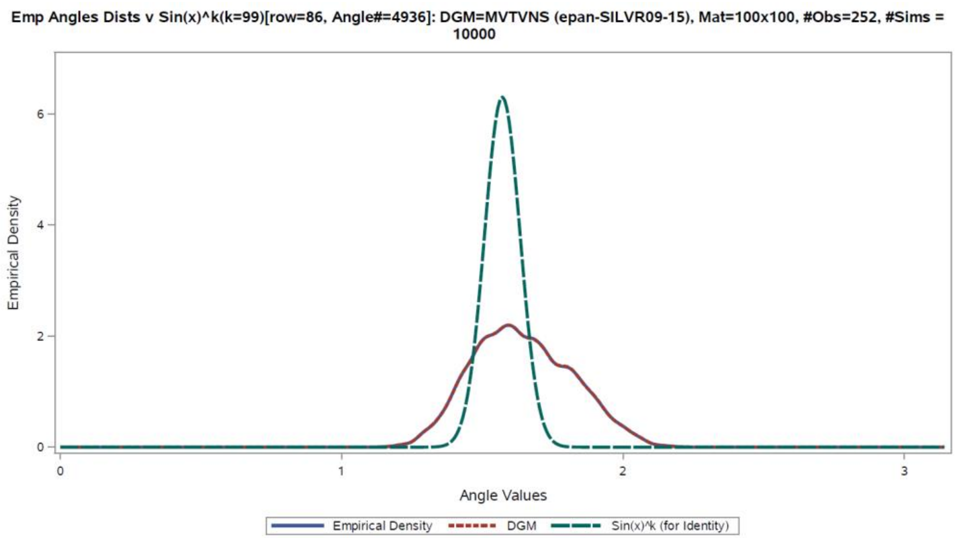
Angle Distributions

n = 101



PRNG Seed = 12345
RunID = Gs5MXMp2

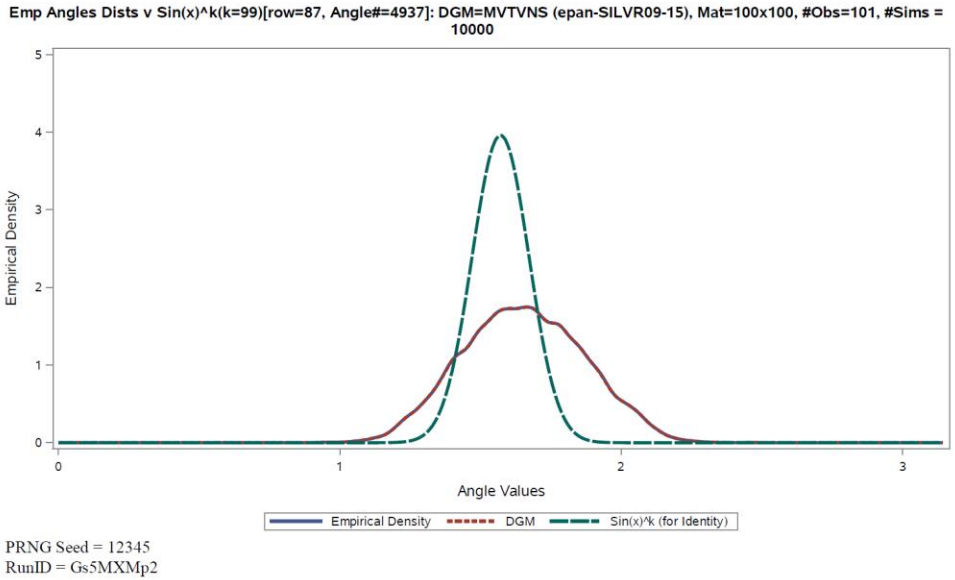
n = 252



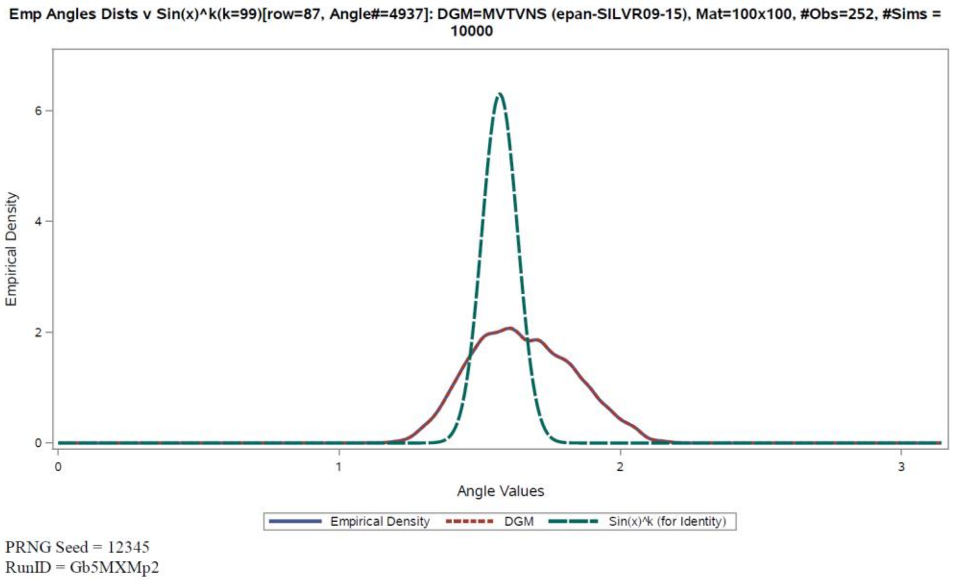
PRNG Seed = 12345
RunID = Gb5MXMp2

Angle Distributions

n = 101

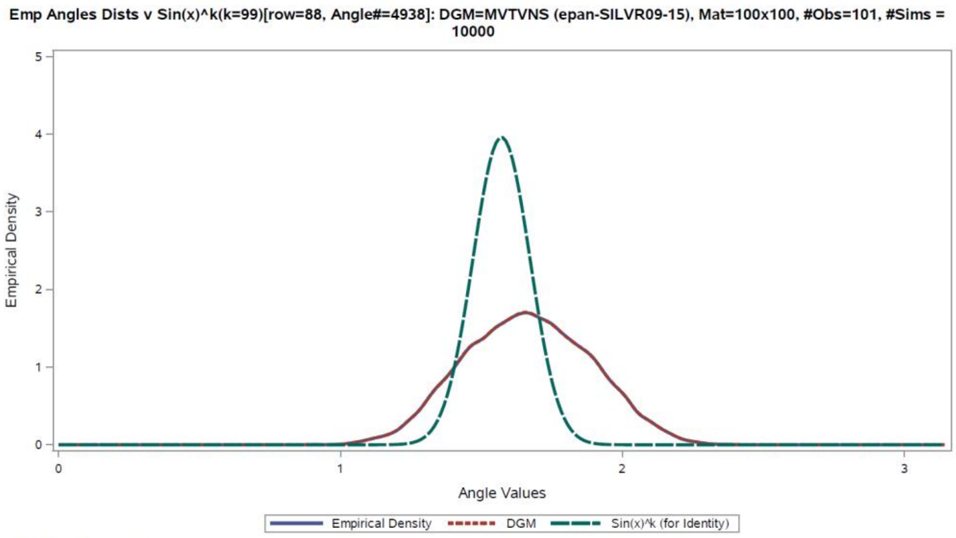


n = 252



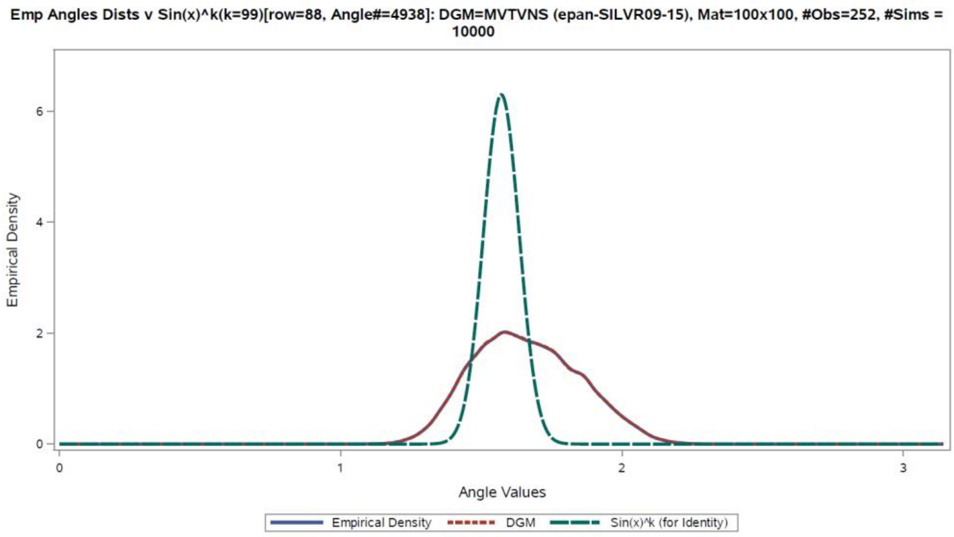
Angle Distributions

n = 101



PRNG Seed = 12345
RunID = Gs5MXMp2

n = 252

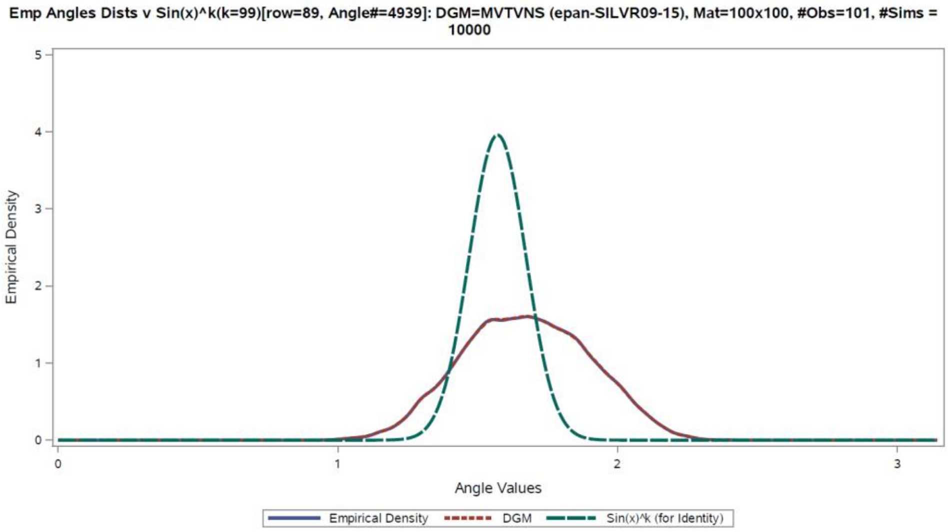


PRNG Seed = 12345
RunID = Gb5MXMp2

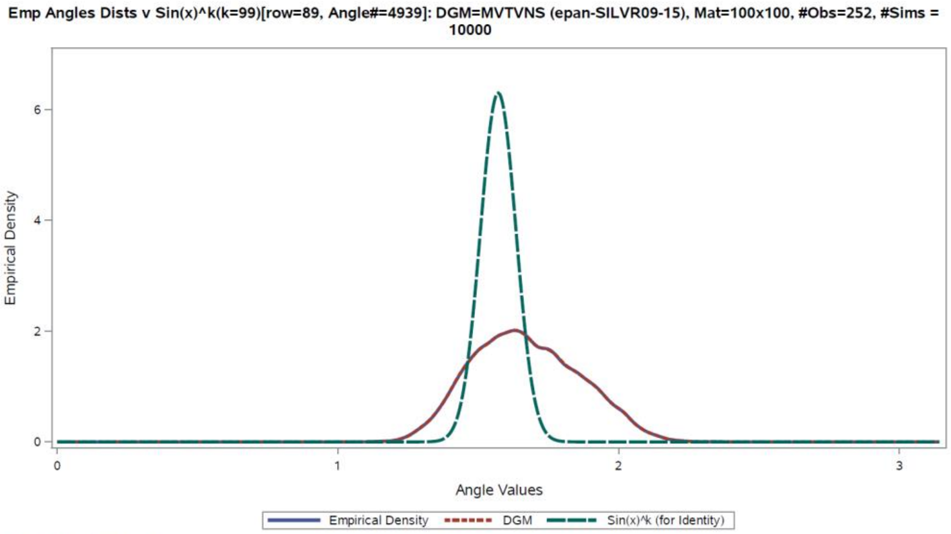
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Angle Distributions

n = 101 | **n = 252**



PRNG Seed = 12345
RunID = Gs5MXMp2

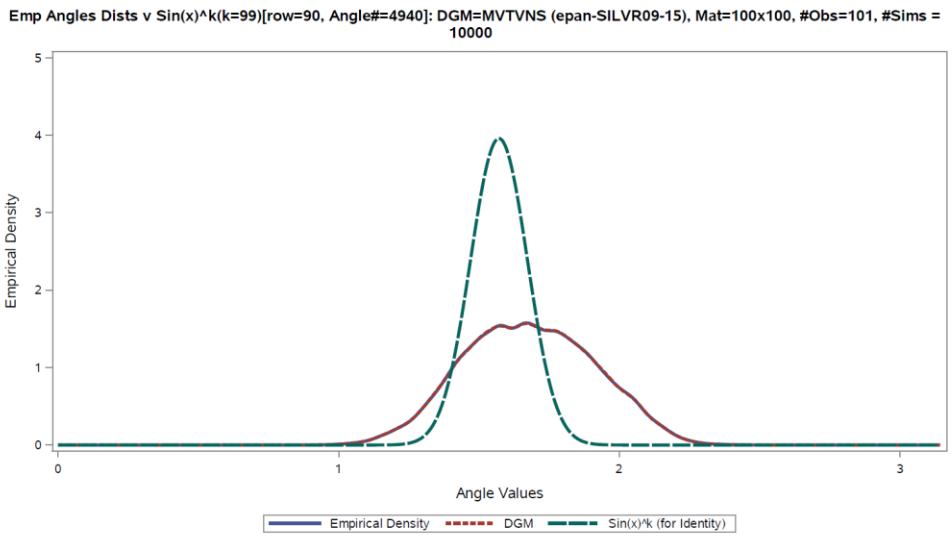


PRNG Seed = 12345
RunID = Gb5MXMp2

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Angle Distributions

n = 101 **n = 252**



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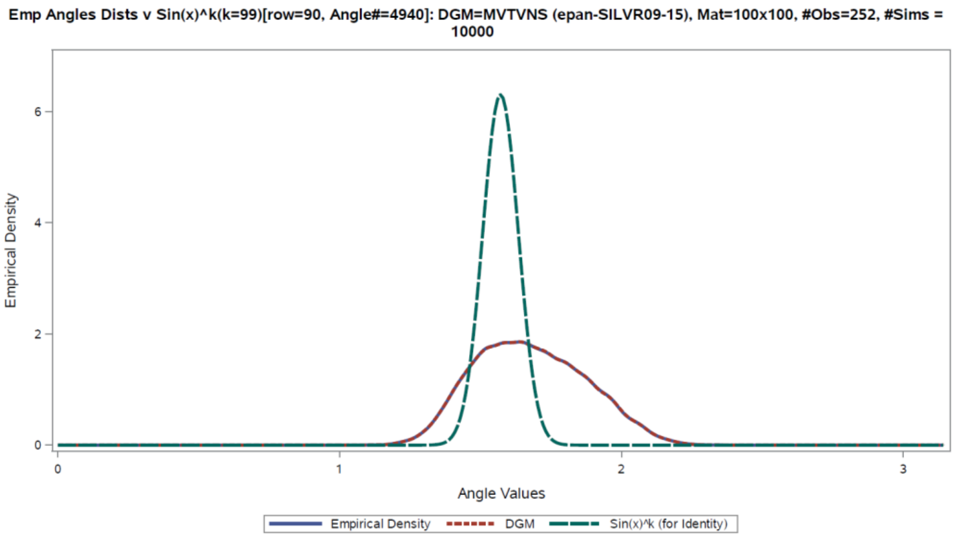


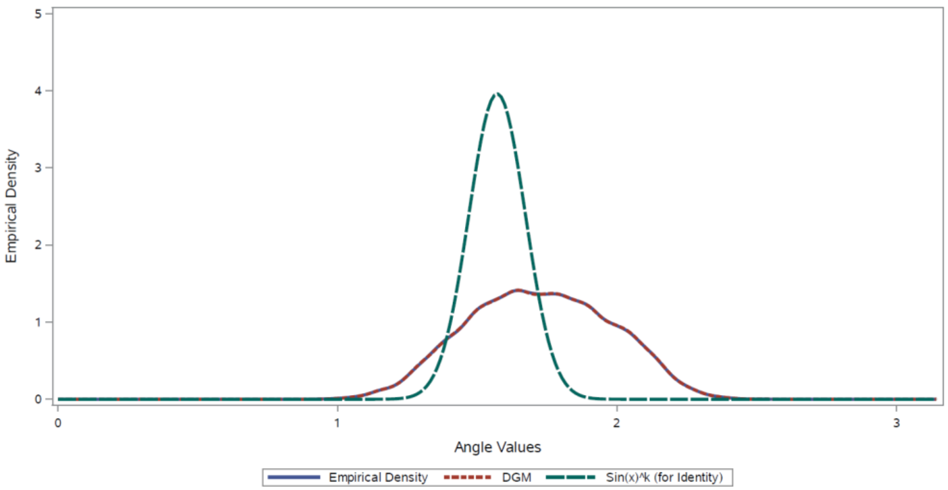
PRNG Seed = 12345
RunID = Gb5MXMp2

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Angle Distributions

n = 101

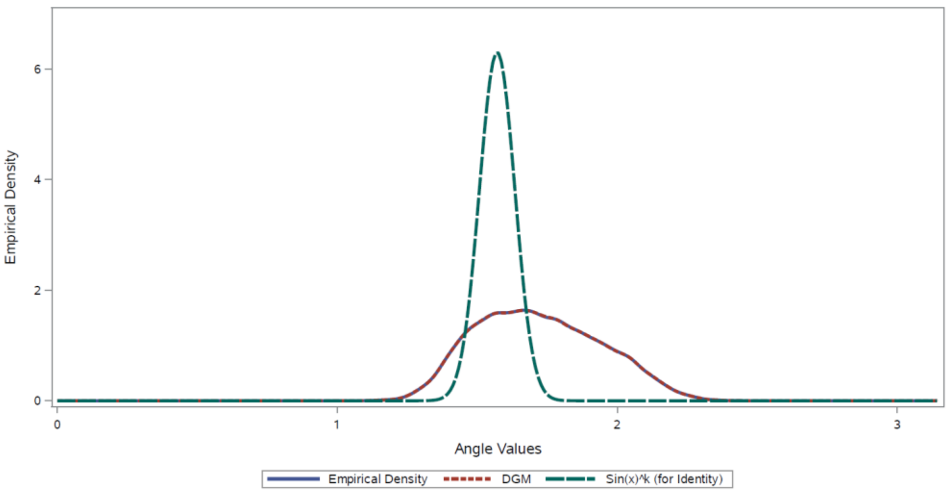
Emp Angles Dists v Sin(x)^k(k=99)[row=96, Angle#=4946]: DGM=MVTVNS (epan-SILVR09-15), Mat=100x100, #Obs=101, #Sims = 10000



PRNG Seed = 12345
RunID = Gs5MXMp2

n = 252

Emp Angles Dists v Sin(x)^k(k=99)[row=96, Angle#=4946]: DGM=MVTVNS (epan-SILVR09-15), Mat=100x100, #Obs=252, #Sims = 10000



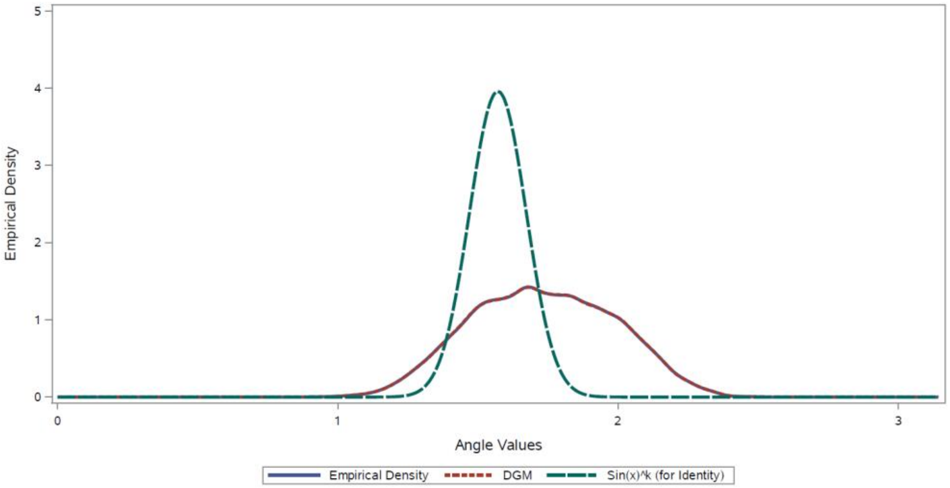
PRNG Seed = 12345
RunID = Gb5MXMp2

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Angle Distributions

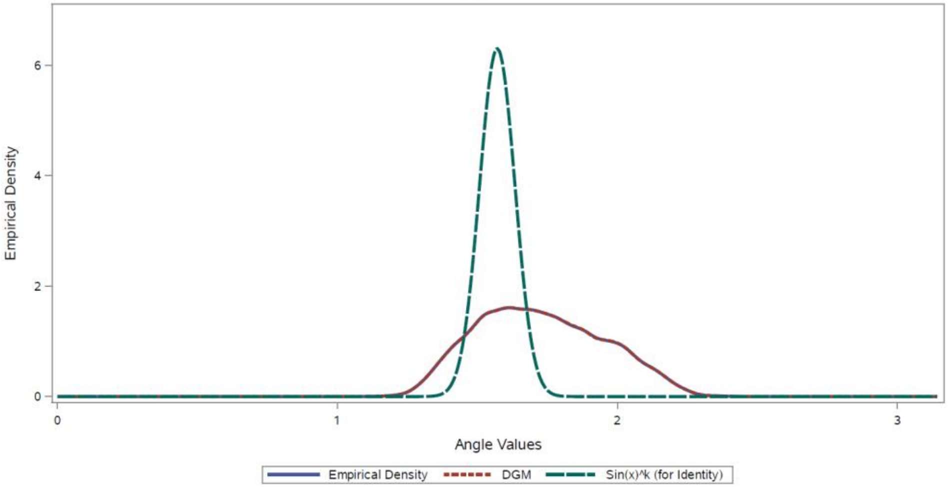
n = 101 **n = 252**

Emp Angles Dists v Sin(x)^k(k=99)[row=97, Angle#=4947]: DGM=MVTVNS (epan-SILVR09-15), Mat=100x100, #Obs=101, #Sims = 10000



PRNG Seed = 12345
RunID = Gs5MXMp2

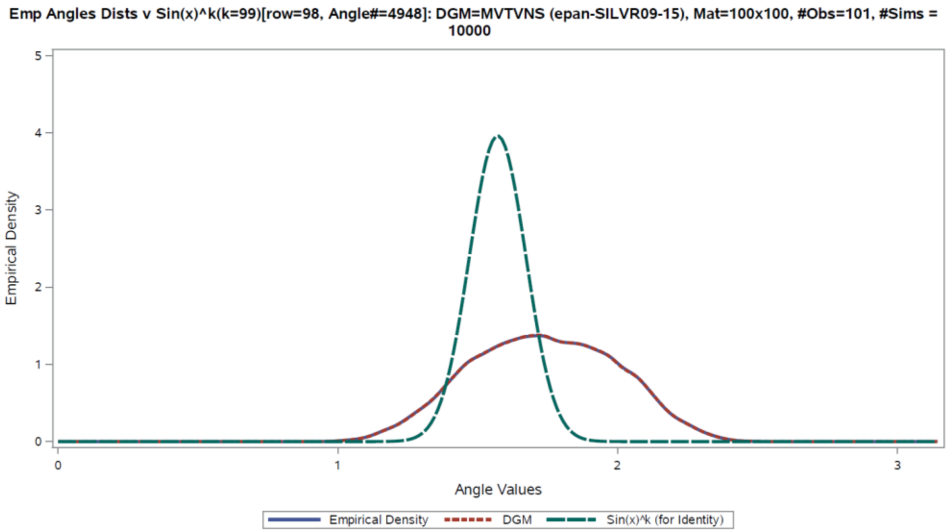
Emp Angles Dists v Sin(x)^k(k=99)[row=97, Angle#=4947]: DGM=MVTVNS (epan-SILVR09-15), Mat=100x100, #Obs=252, #Sims = 10000



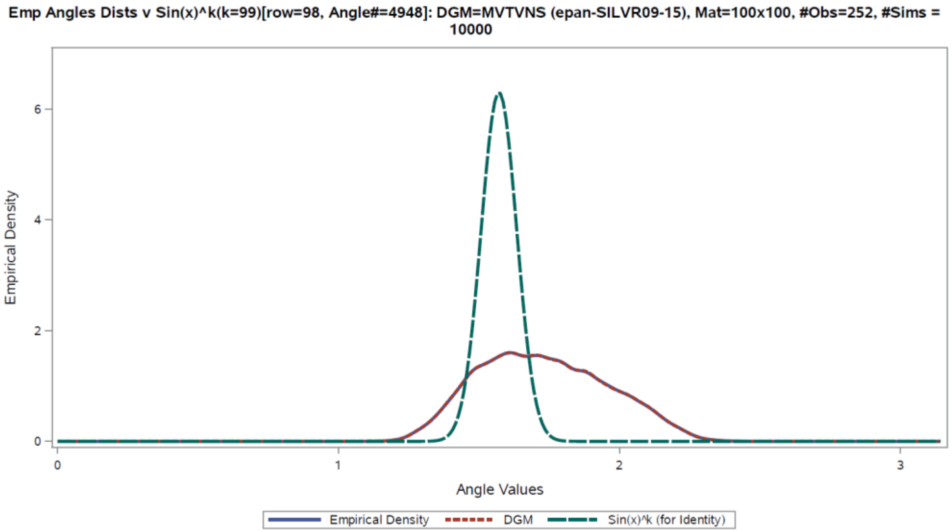
PRNG Seed = 12345
RunID = Gb5MXMp2

Angle Distributions

n = 101 | **n = 252**



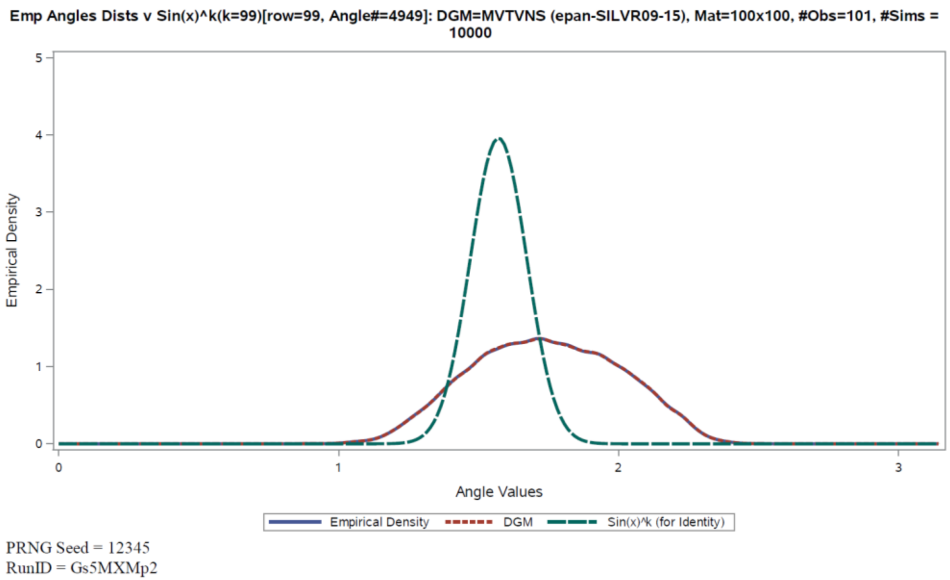
PRNG Seed = 12345
RunID = Gs5MXMp2



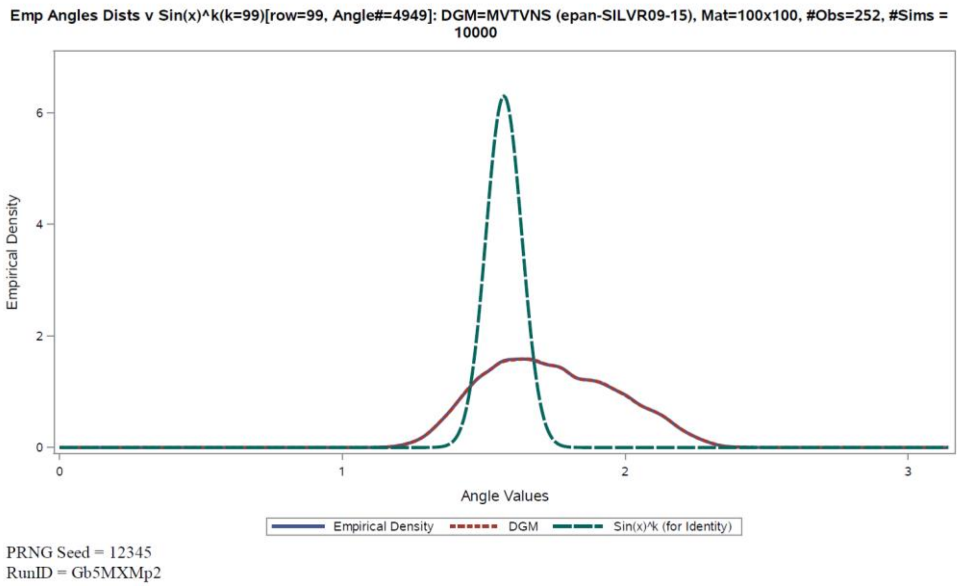
PRNG Seed = 12345
RunID = Gb5MXMp2

Angle Distributions

n = 101



n = 252

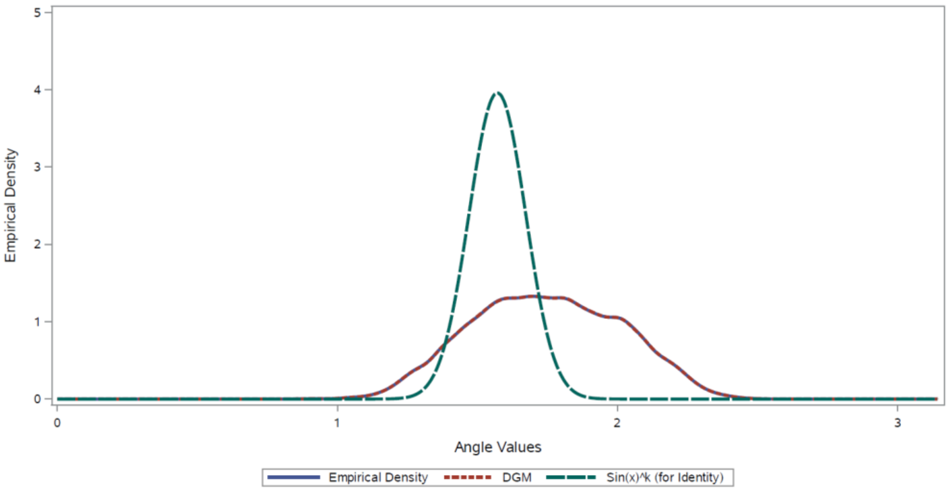


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case G

Angle Distributions

n = 101

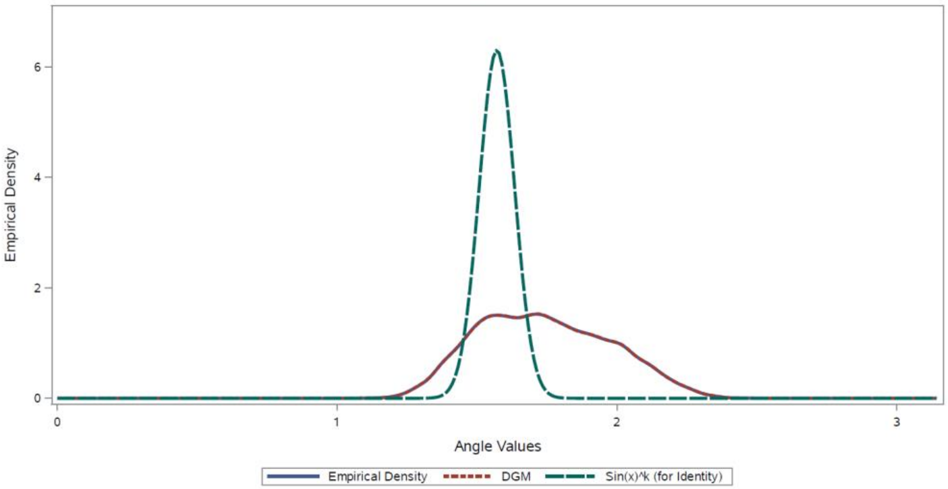
Emp Angles Dists v Sin(x)^k(k=99)[row=100, Angle#=4950]: DGM=MVTVNS (epan-SILVR09-15), Mat=100x100, #Obs=101, #Sims = 10000



PRNG Seed = 12345
RunID = Gs5MXMp2

n = 252

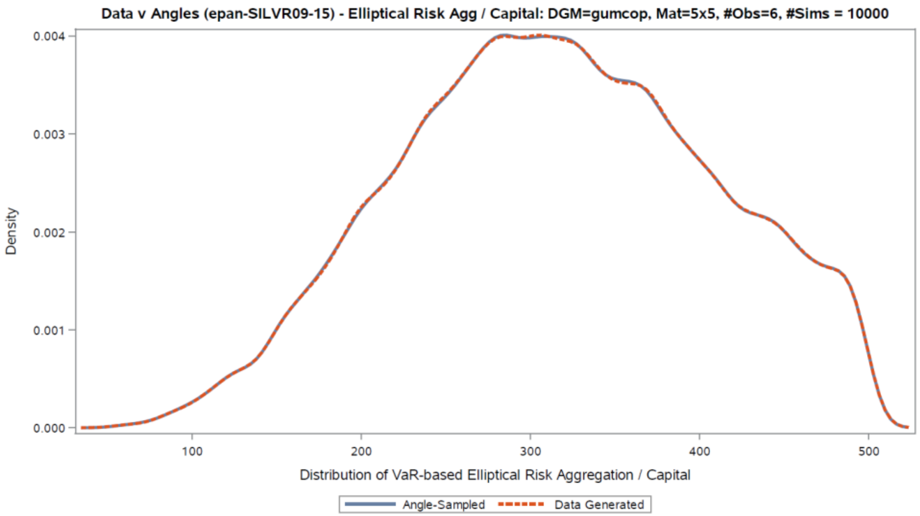
Emp Angles Dists v Sin(x)^k(k=99)[row=100, Angle#=4950]: DGM=MVTVNS (epan-SILVR09-15), Mat=100x100, #Obs=252, #Sims = 10000



PRNG Seed = 12345
RunID = Gb5MXMp2

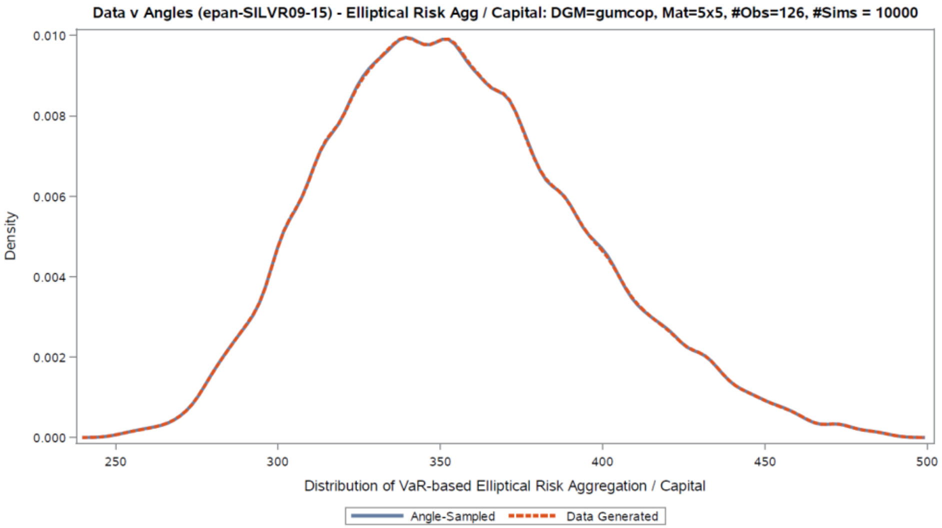
Elliptical Capital

n = 6



PRNG Seed = 12345
RunID = Hns5x5GUM

n = 126

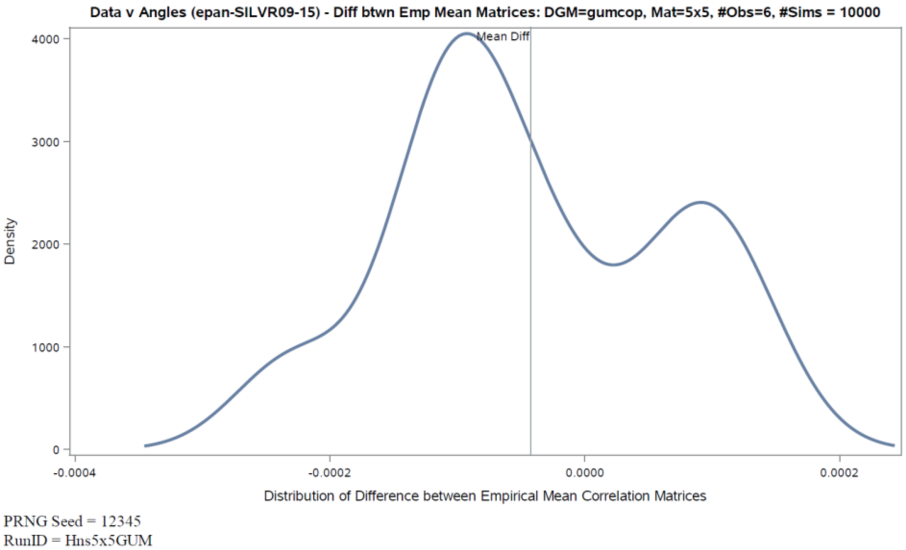


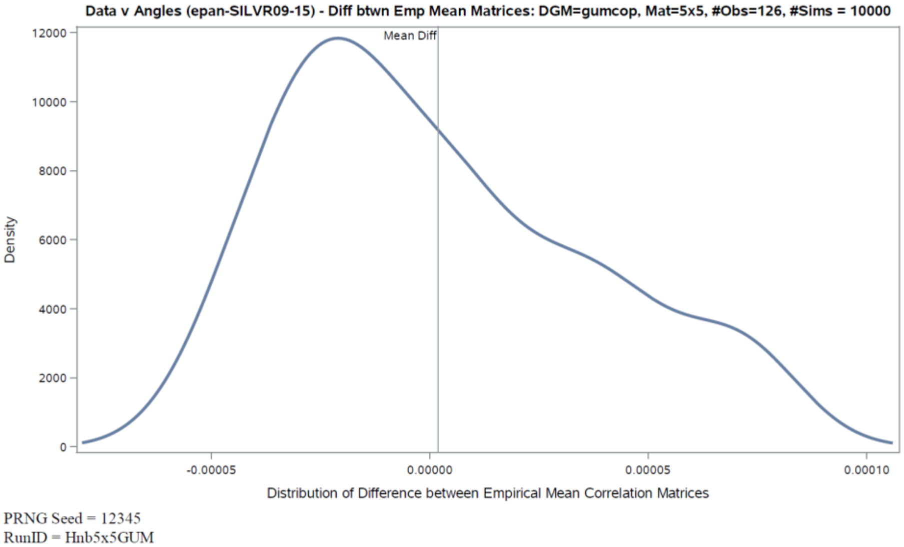
PRNG Seed = 12345
RunID = Hnb5x5GUM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

Difference Between Mean Empirical Matrices

n = 6 **n = 126**

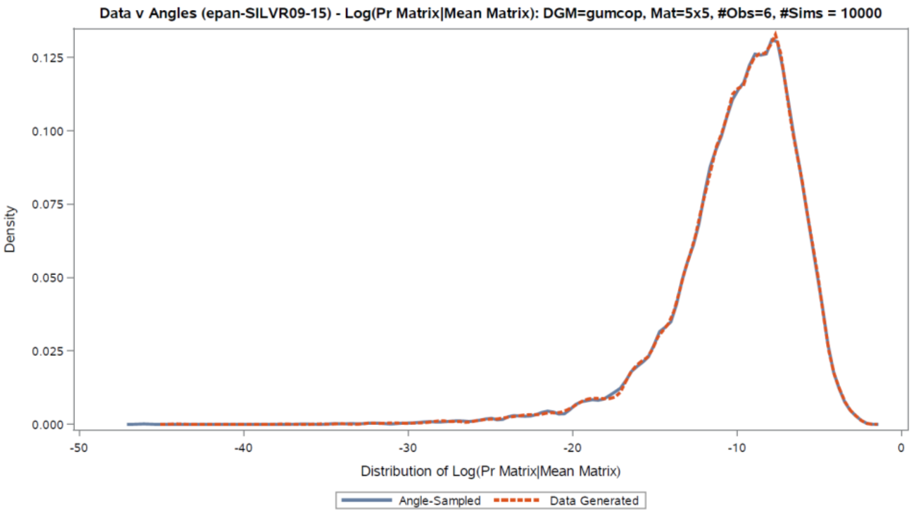




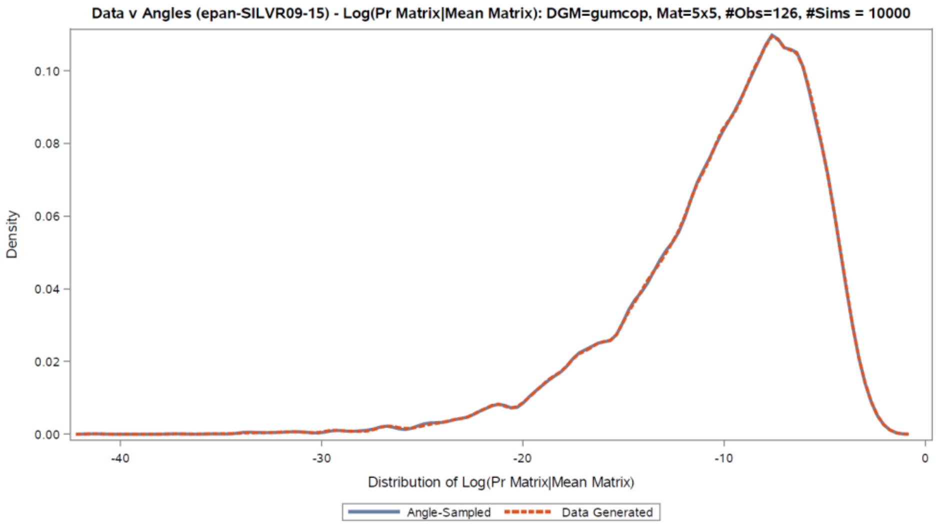
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

LNP

n = 6 **n = 126**



PRNG Seed = 12345
RunID = Hns5x5GUM

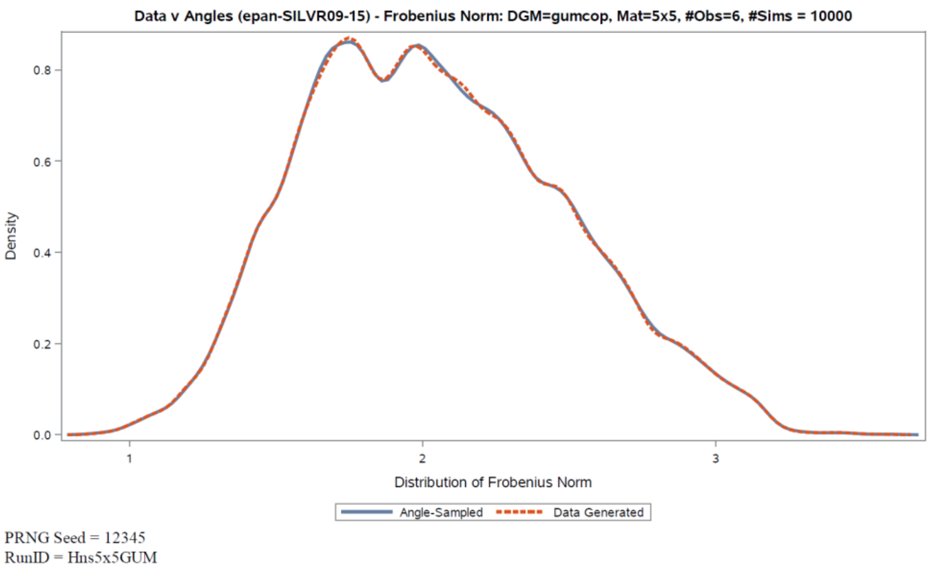


PRNG Seed = 12345
RunID = Hnb5x5GUM

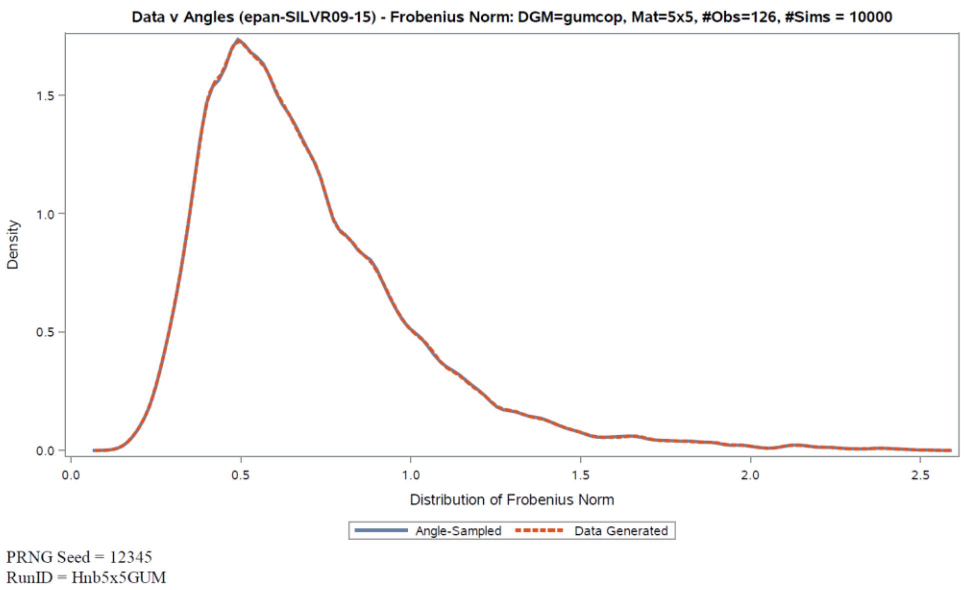
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

Euclidian/Frobenius Norm

n = 6

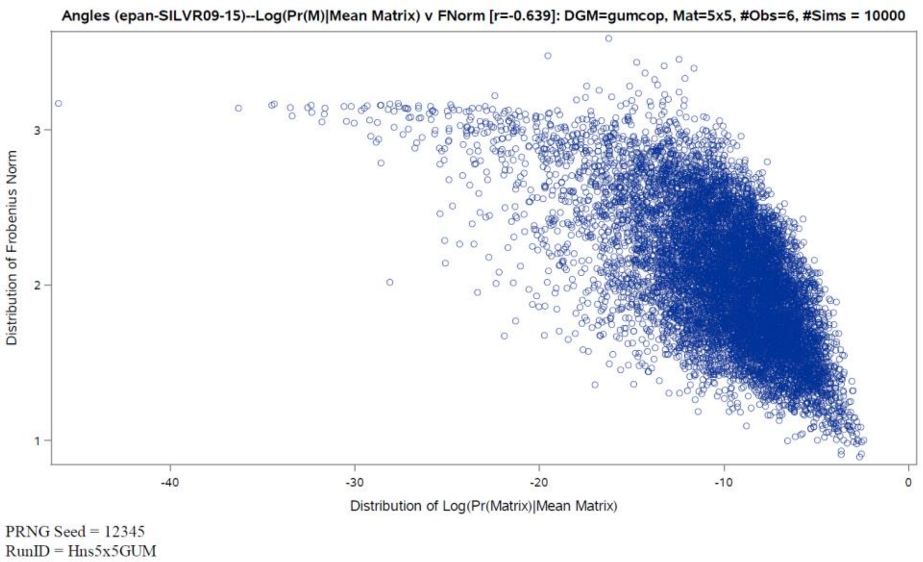


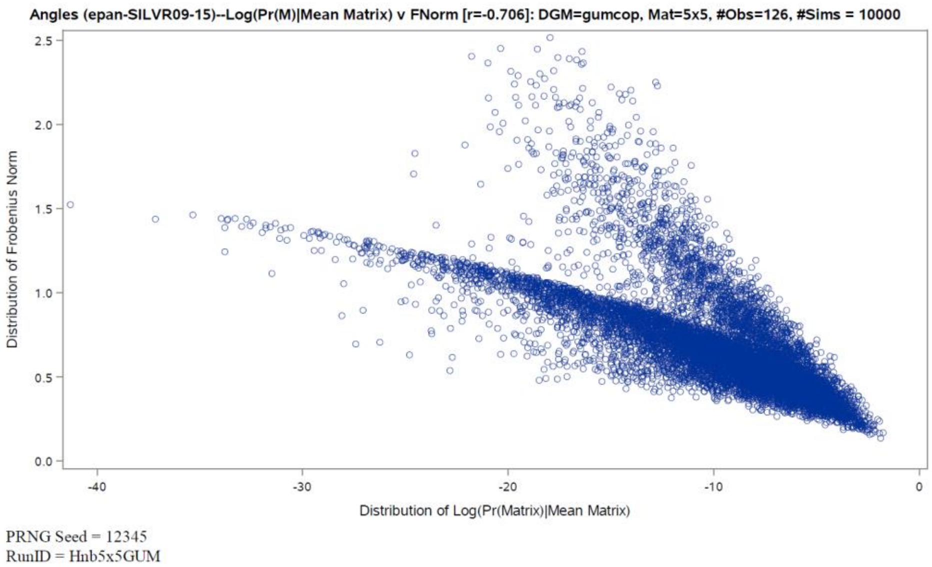
n = 126



LNP v Euclidian/Frobenius Norm

n = 6 | **n = 126**

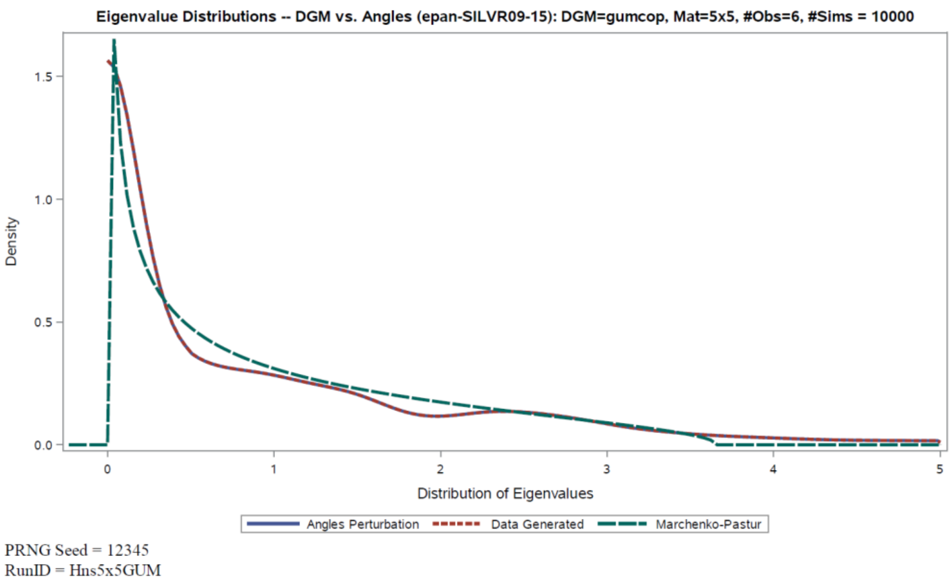




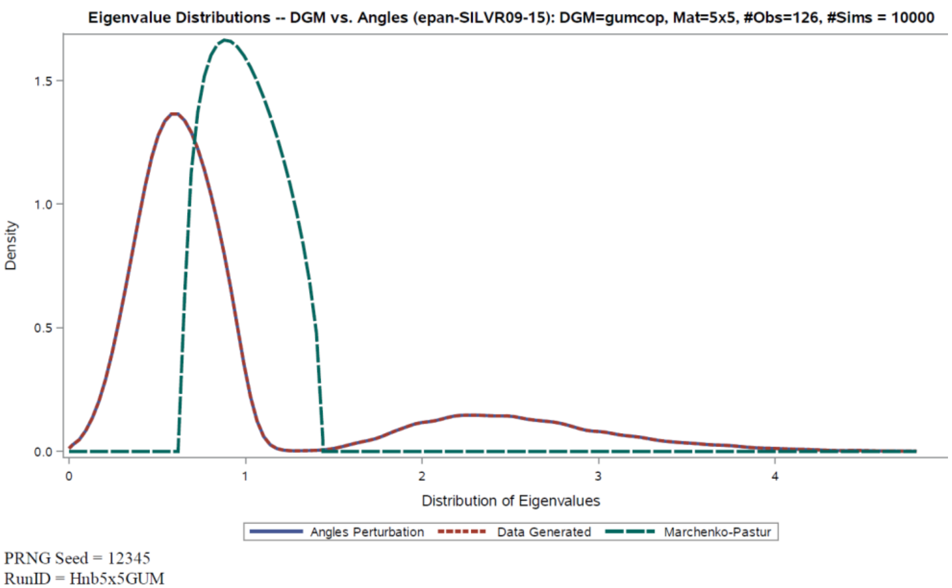
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

Spectral Distributions

n = 6

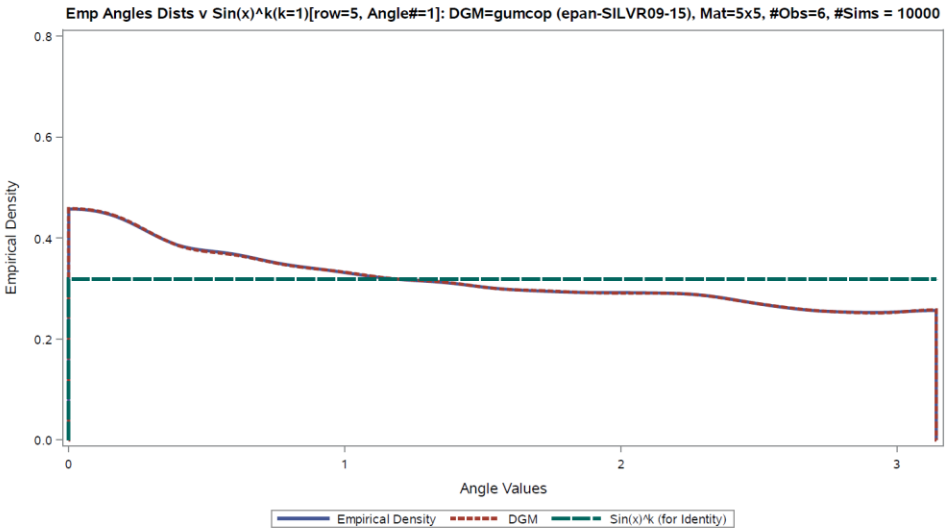


n = 126



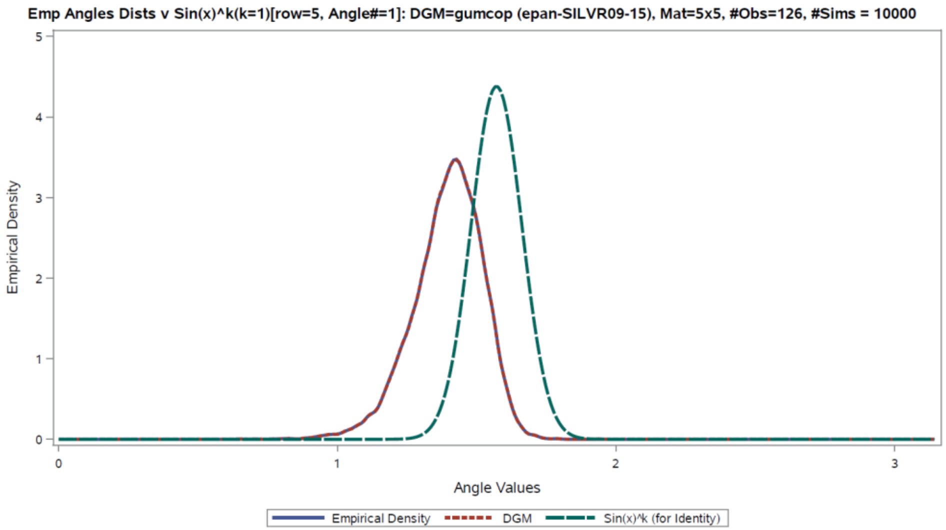
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Hns5x5GUM

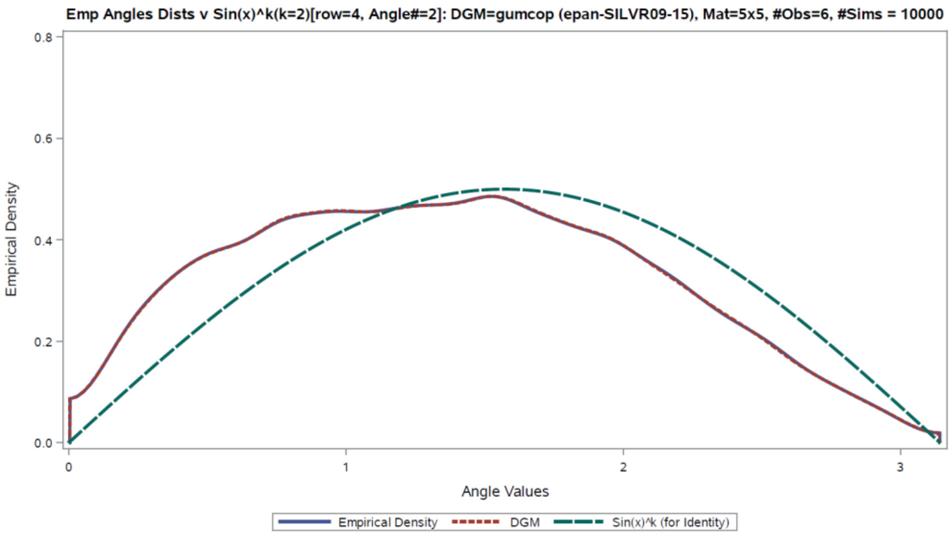
n = 126



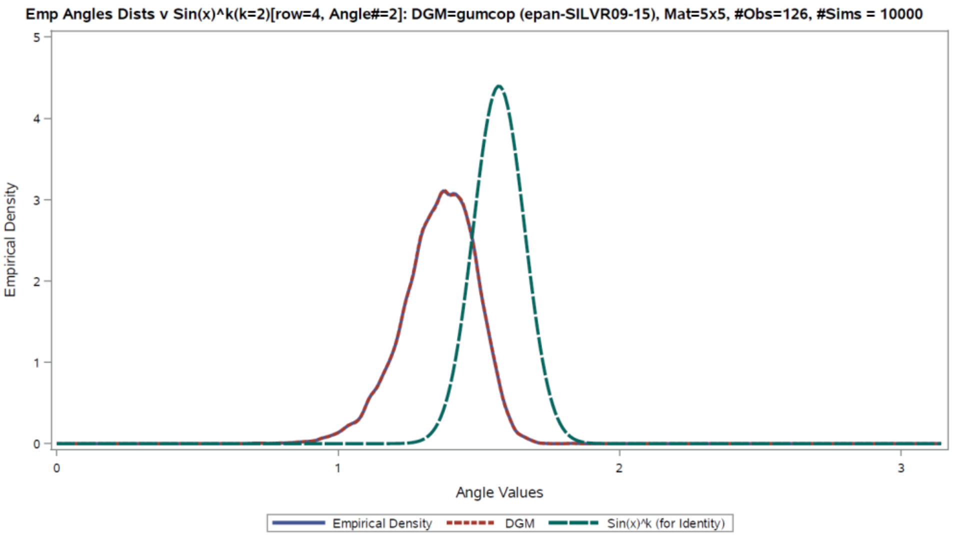
PRNG Seed = 12345
RunID = Hnb5x5GUM

Angle Distributions

n = 6 **n = 126**



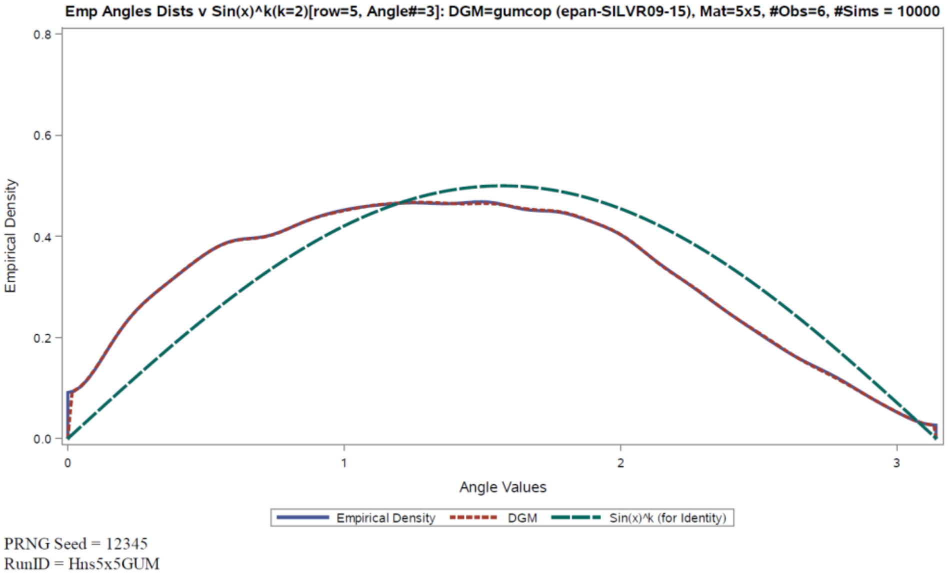
PRNG Seed = 12345
RunID = Hns5x5GUM

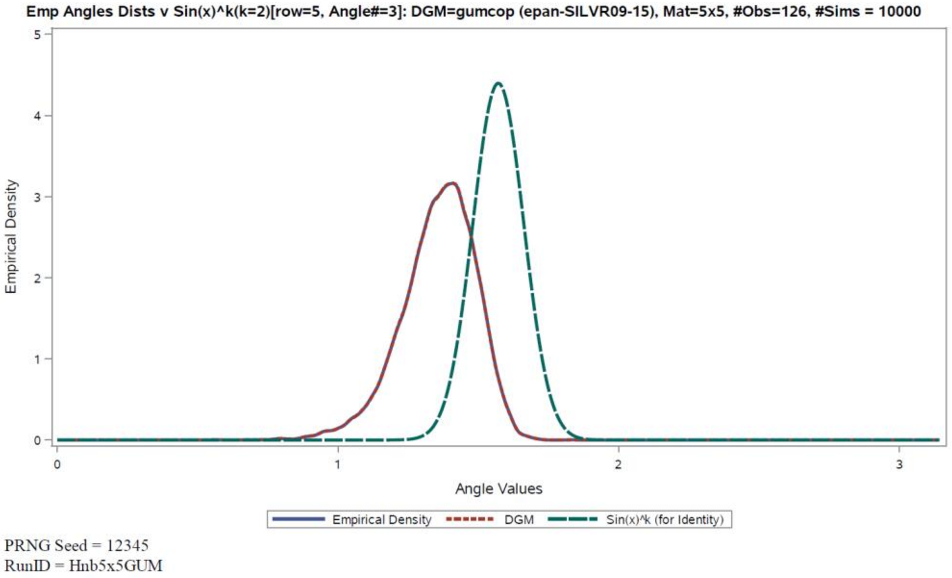


PRNG Seed = 12345
RunID = Hnb5x5GUM

Angle Distributions

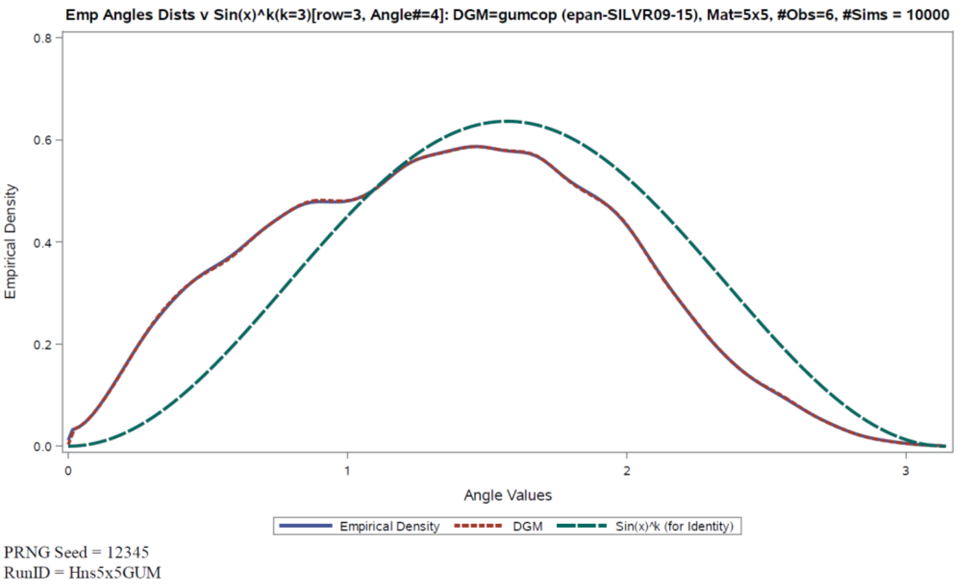
n = 6 | **n = 126**



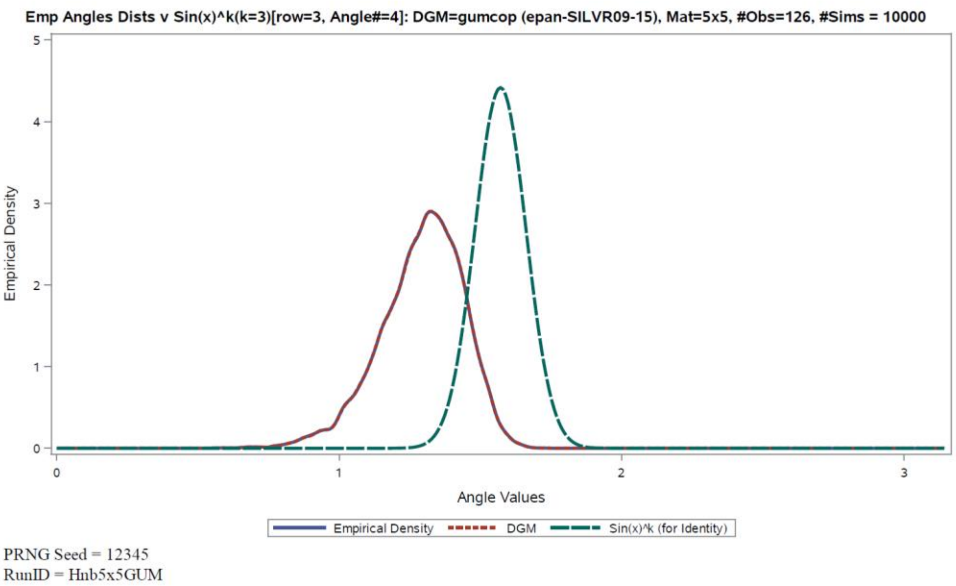


Angle Distributions

n = 6



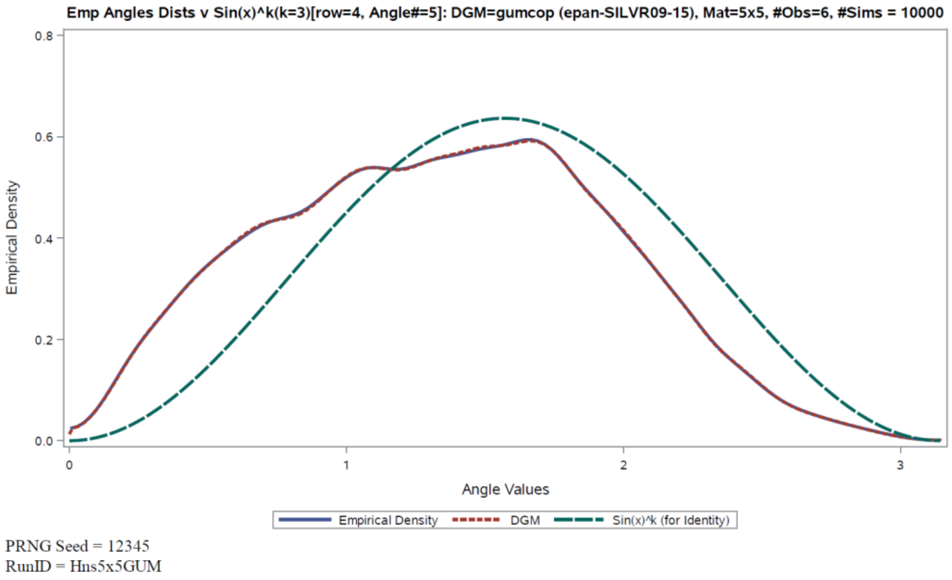
n = 126

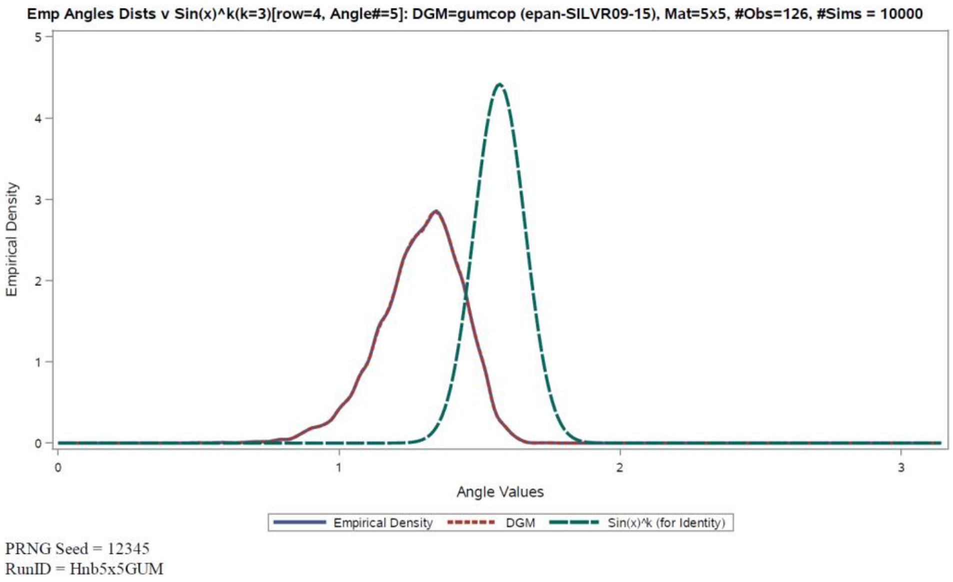


XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

Angle Distributions

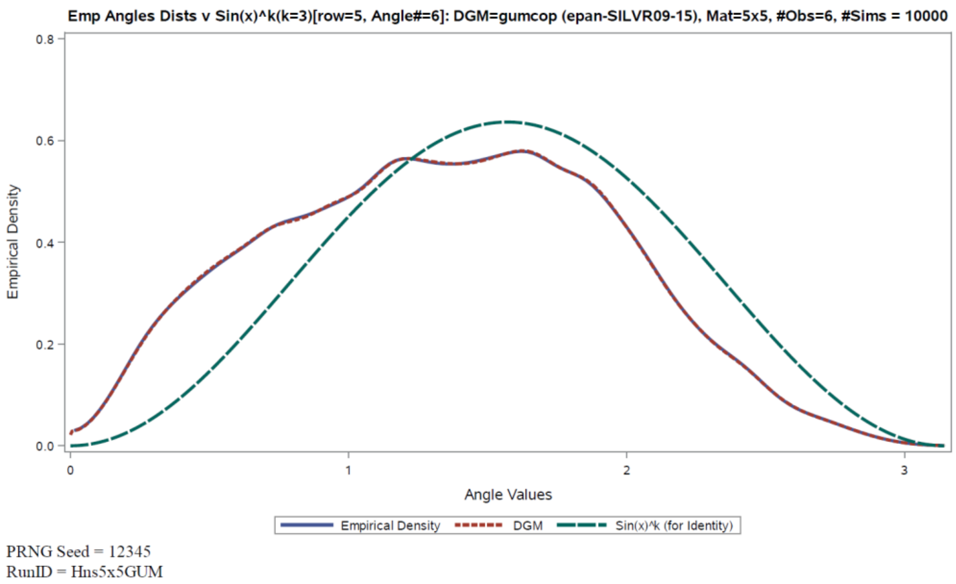
n = 6 **n = 126**



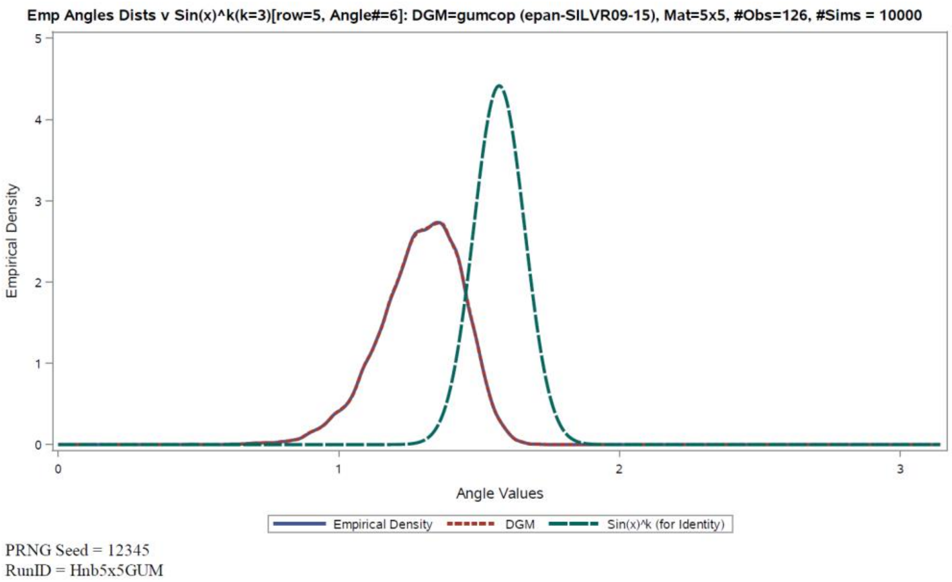


Angle Distributions

n = 6



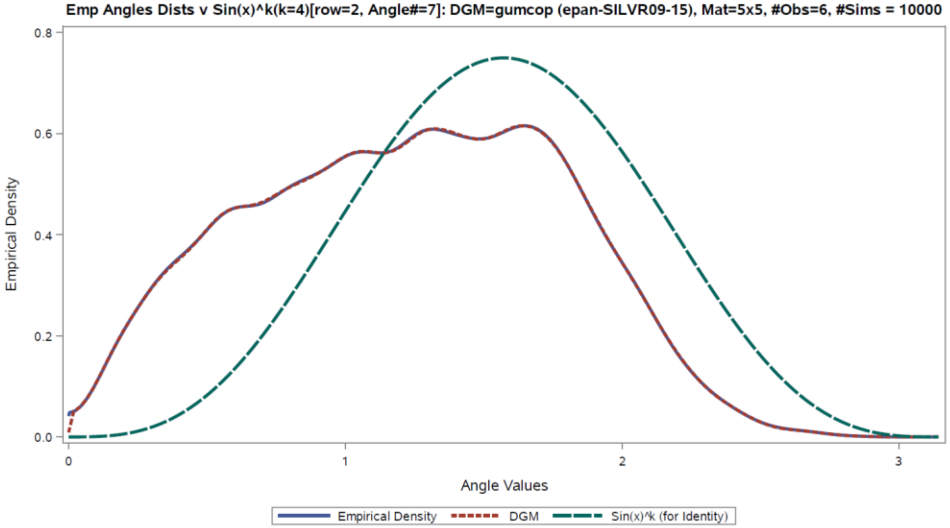
n = 126



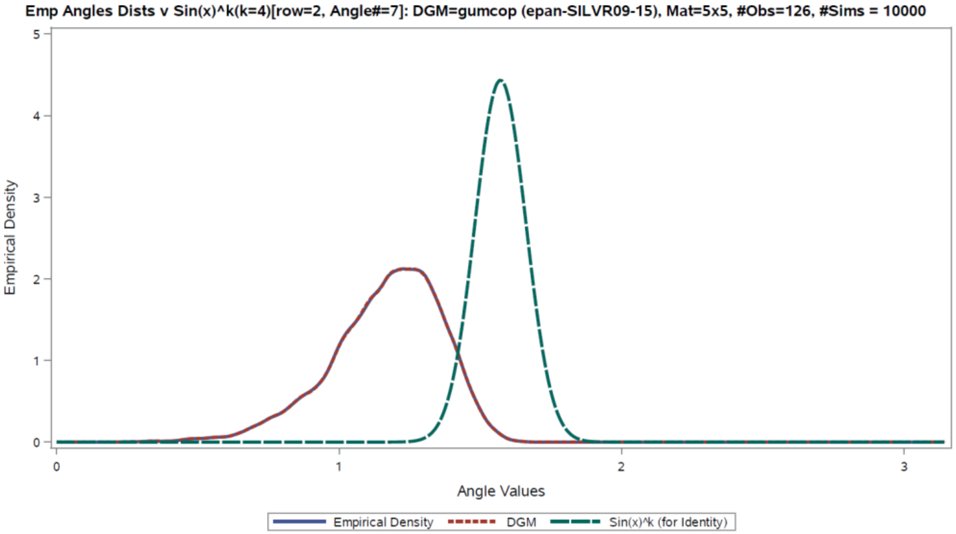
XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

Angle Distributions

n = 6 | **n = 126**



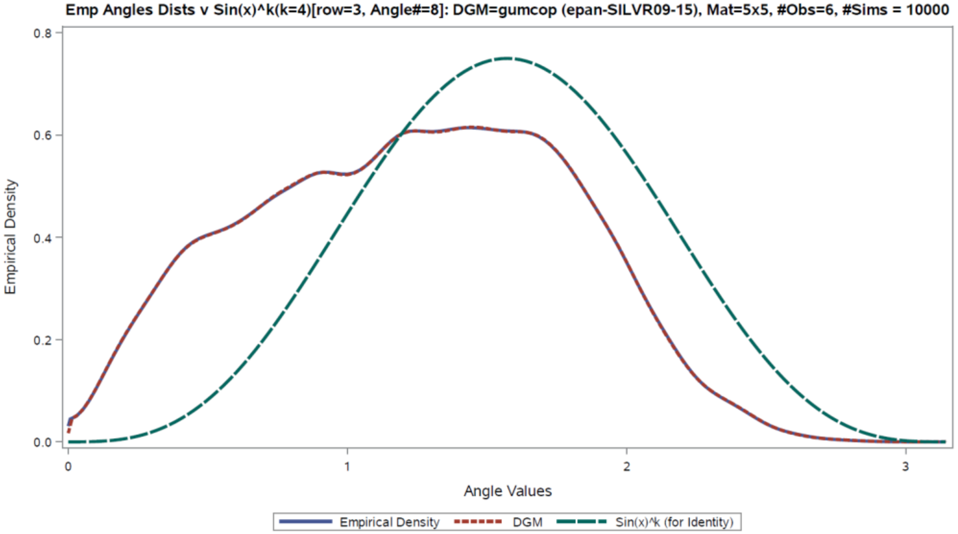
PRNG Seed = 12345
RunID = Hns5x5GUM



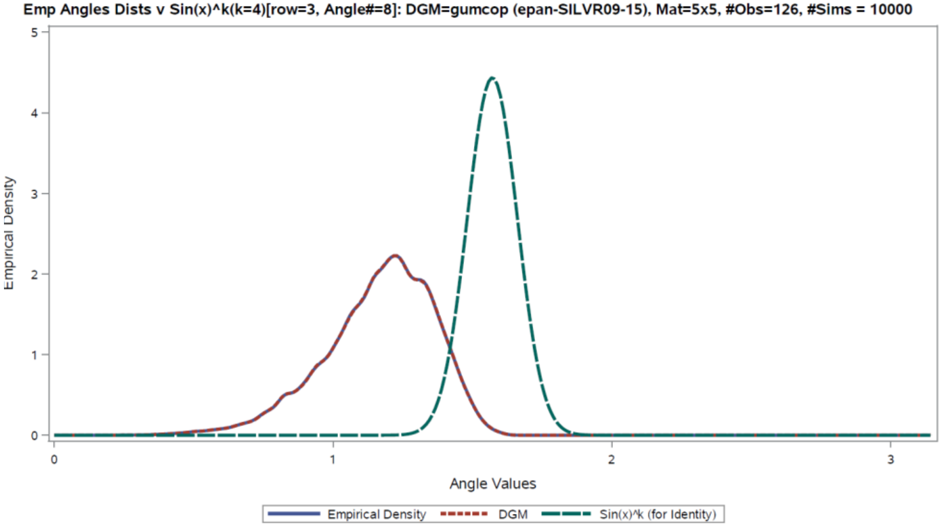
PRNG Seed = 12345
RunID = Hnb5x5GUM

Angle Distributions

n = 6 **n = 126**



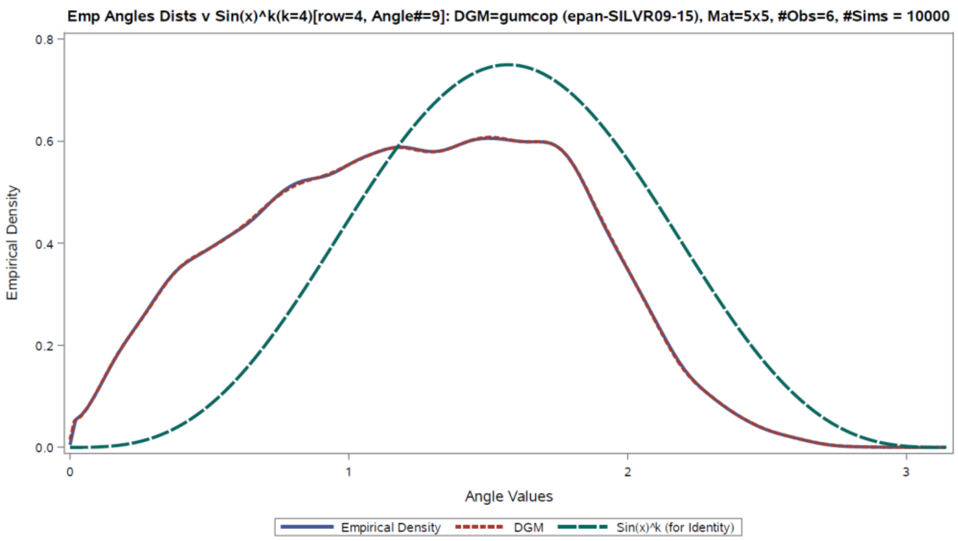
PRNG Seed = 12345
RunID = Hns5x5GUM



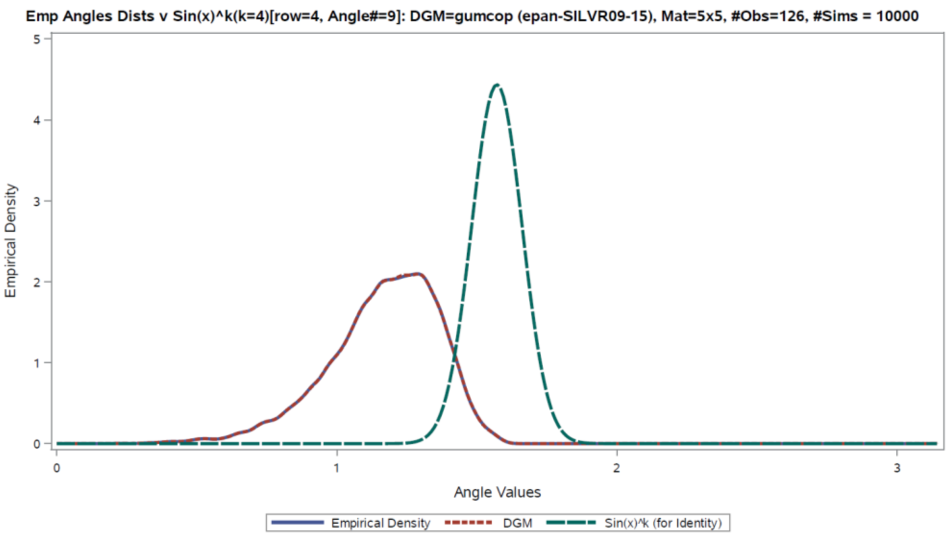
PRNG Seed = 12345
RunID = Hnb5x5GUM

Angle Distributions

n = 6 | **n = 126**



PRNG Seed = 12345
RunID = Hns5x5GUM

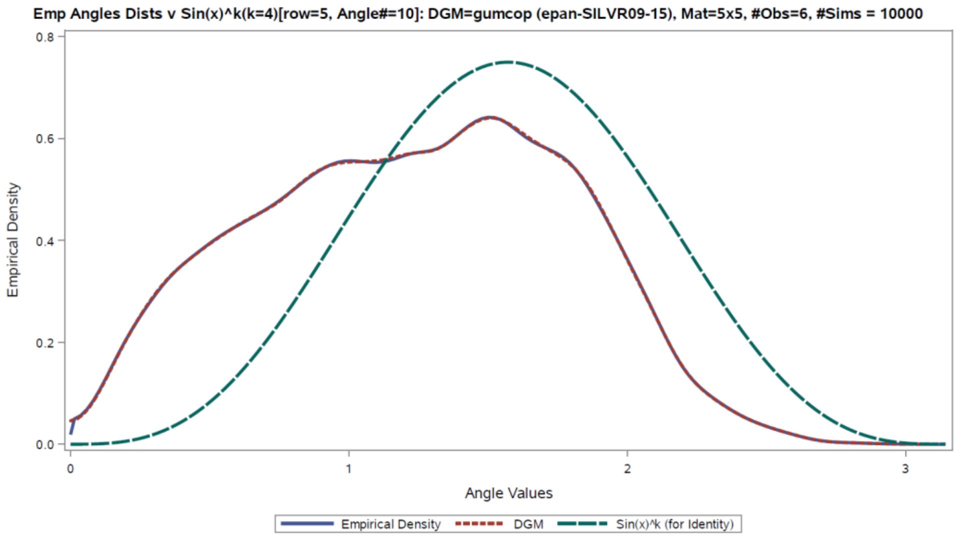


PRNG Seed = 12345
RunID = Hnb5x5GUM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

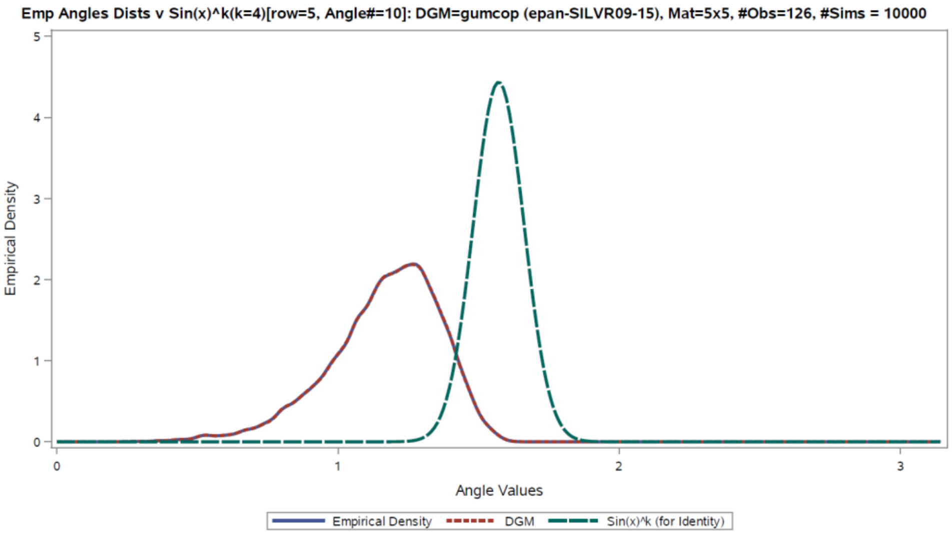
Angle Distributions

n = 6



PRNG Seed = 12345
RunID = Hns5x5GUM

n = 126



PRNG Seed = 12345
RunID = Hnb5x5GUM

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

Correlation Matrices to CDF Matrices

Correlation Matrices | **n=6** | **n=126**

1	1	1	1	1												
0	0.1	0.2	0.55	0.6		0.69	0.64	0.58	0.35	0.32		0.9980	0.9693	0.8607	0.1698	0.1173
0	0.1	0.2	0.55	0.6		0.70	0.63	0.57	0.35	0.32		0.9979	0.9706	0.8614	0.1766	0.1199
0	0.1	0.2	0.55	0.6		0.70	0.63	0.57	0.35	0.32		0.9985	0.9728	0.8620	0.1744	0.1194
0	0.1	0.2	0.55	0.6		0.69	0.63	0.56	0.35	0.32		0.9988	0.9715	0.8625	0.1735	0.1206
1	1	1	1	1												
0	0.1	0.2	0.55	0.3		0.63	0.57	0.53	0.42	0.68		0.9897	0.9207	0.7681	0.2606	0.9997
0	0.1	0.2	0.55	0.3		0.63	0.57	0.53	0.42	0.68		0.9887	0.9177	0.7701	0.2656	0.9998
0	0.1	0.2	0.55	0.3		0.62	0.57	0.53	0.42	0.67		0.9895	0.9184	0.7621	0.2647	0.9997
1	1	1	1	1												
0	0.1	0.2	0.55	0.6		0.60	0.56	0.53	0.47	0.42		0.9721	0.8485	0.6819	0.3200	0.1070
0	0.1	0.2	0.55	0.6		0.59	0.55	0.52	0.47	0.41		0.9699	0.8450	0.6800	0.3138	0.1069
1	1	1	1	1												
0	0.1	0.2	0.55	0.4		0.57	0.54	0.53	0.50	0.59		0.9447	0.7838	0.6334	0.3550	0.9959
1	1	1	1	1												

CDF Matrices to Correlation Matrices

CDF Matrices | **n=6** | **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.32	0.48	0.16	0.55	-0.36		0.37	0.42	0.33	0.44	0.17
0.5	0.4	0.6	0.35	0.9		0.32	0.47	0.16	0.55	-0.36		0.37	0.42	0.33	0.44	0.17
0.5	0.4	0.6	0.35	0.9		0.32	0.48	0.15	0.55	-0.36		0.37	0.42	0.33	0.44	0.18
0.5	0.4	0.6	0.35	0.9		0.31	0.47	0.15	0.55	-0.36		0.37	0.42	0.33	0.44	0.17
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.29	0.52	0.07	0.34	-0.14		0.36	0.42	0.31	0.38	0.18
0.5	0.4	0.6	0.6	0.8		0.29	0.53	0.07	0.34	-0.14		0.37	0.42	0.31	0.38	0.18
0.5	0.4	0.6	0.6	0.8		0.29	0.52	0.06	0.33	-0.14		0.36	0.42	0.31	0.38	0.18
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.32	0.61	0.03	0.59	0.68		0.36	0.42	0.30	0.42	0.31
0.5	0.4	0.6	0.4	0.3		0.30	0.60	0.00	0.58	0.68		0.36	0.42	0.30	0.41	0.31
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.35	0.73	-0.09	0.45	-0.01		0.35	0.43	0.29	0.39	0.12
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 1: Empirical Results of NAbC, Full Matrix – Case H

CDF %Shift Matrices to Correlation Matrices

CDF %Shift Matrices — **n=6** — **n=126**

					lnMatPr	-2.279	-2.185	-7.124	-6.748	-13.519		-2.008	-2.460	-6.069	-7.882	-10.321
					FNorm	1.103	1.096	2.303	2.172	2.661		0.234	0.263	0.649	0.649	0.950
					Rnk_lnMat	2	1	4	3	5		1	2	3	4	5
					Rnk_FNorr	2	1	4	3	5		1	2	3	4	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.46	0.13	0.69	-0.10	0.76		0.43	0.34	0.51	0.27	0.54
20	-20	50	-50	60		0.45	0.13	0.69	-0.10	0.76		0.43	0.34	0.51	0.27	0.54
20	-20	50	-50	60		0.45	0.13	0.68	-0.11	0.76		0.43	0.34	0.51	0.26	0.54
20	-20	50	-50	60		0.45	0.12	0.68	-0.11	0.76		0.43	0.34	0.51	0.27	0.54
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.51	0.07	0.81	-0.20	0.91		0.44	0.33	0.53	0.24	0.59
20	-20	50	-50	70		0.52	0.07	0.81	-0.19	0.91		0.44	0.33	0.53	0.24	0.59
20	-20	50	-50	70		0.51	0.05	0.81	-0.21	0.91		0.44	0.33	0.53	0.24	0.59
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.59	0.01	0.90	-0.26	0.98		0.45	0.32	0.55	0.22	0.64
20	-20	50	-50	80		0.59	0.00	0.90	-0.27	0.98		0.45	0.32	0.55	0.22	0.64
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.70	-0.14	0.97	-0.34	1.00		0.45	0.31	0.57	0.20	0.70
						1	1	1	1	1		1	1	1	1	1

XI. APPENDIX 2: Empirical Results of NAbC, Targeted Scenario Matrix

- <u>Empirical Results from NAbC – Example Cases, Targeted Scenario versions</u>:

A. **TS**: MVG – Gaussian Identity Matrix: Targeted Scenario with only first 5 (of 10) cells targeted (n=6, n=126)

F. **TS**: MVTVNS: Targeted Scenario with only first 5 (of 10) cells targeted ($\boldsymbol{\rho}$ = block structure, df=3,4,5,6,7, skewness=1,0.6,0,-0.6,-1, autocorrelation=-0.25,0,0.25,0.5,0.75, nonstationarity=3σ, σ/3, σ, n/3 each) (n=6, n=126)

Case F.TS: *R* =

1	-0.1	-0.1	0.2	0.2
-0.1	1	-0.1	0.2	0.2
-0.1	-0.1	1	0.2	0.2
0.2	0.2	0.2	1	0.5
0.2	0.2	0.2	0.5	1

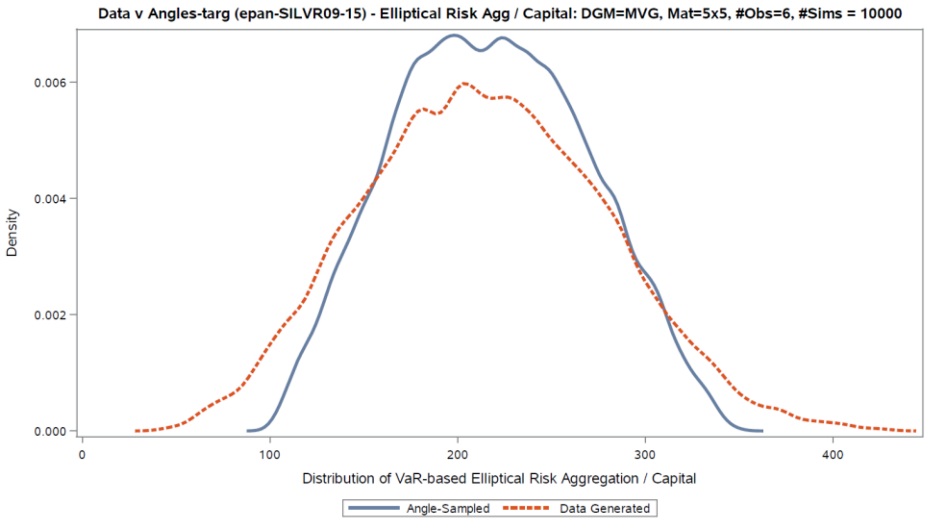
- <u>Empirical Results from NAbC – Summary</u>:

- <u>Case A.TS</u>: shows the effects of imposing targeted scenarios wherein only specified cells (cells 1-5) are allowed to vary. Note that the tabular results show that the values of targeted cells are essentially identical to those of Case A., as they should be, while the non-targeted cells (cells 6-10) all remain untouched (see the spikes in these angles' distributions, which are a constant value): they are the cdf's/correlation values associated with the mean empirical correlation matrix (which will be the same as the specified one under elliptical data); yet all the simulated correlation matrices that result from converting the angles matrices to Cholesky factorizations to correlation matrices all are positive definite, as required, because we remain on the hyper hemisphere by sampling angles.

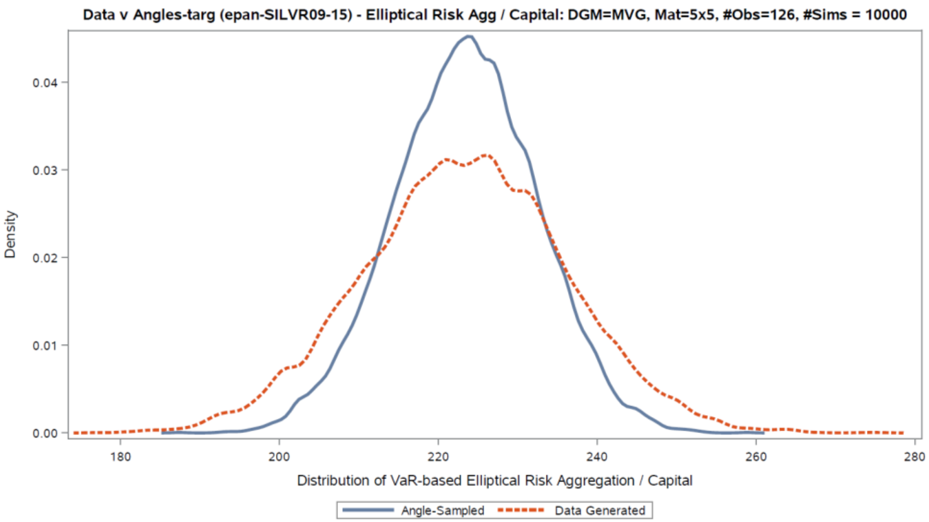
- <u>Case F.TS</u>: shows the same results as Case A.TS under more real world data conditions. The relative effects of imposing a targeted scenario compared to Case F., the unconstrained case, are the same as those when Case A.TS is compared to Case A.

Elliptical Capital – Targeted Scenario v Unconstrained

n = 6 **n = 126**



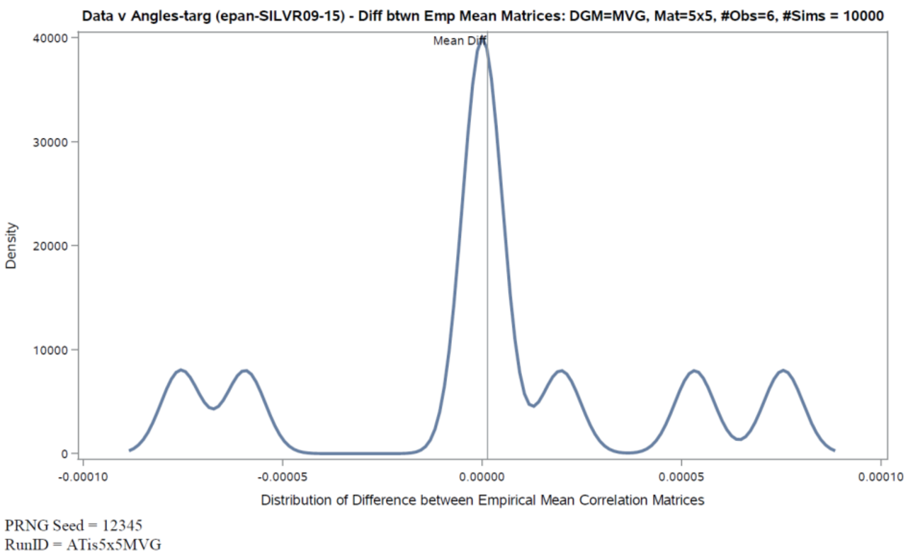
PRNG Seed = 12345
RunID = ATis5x5MVG

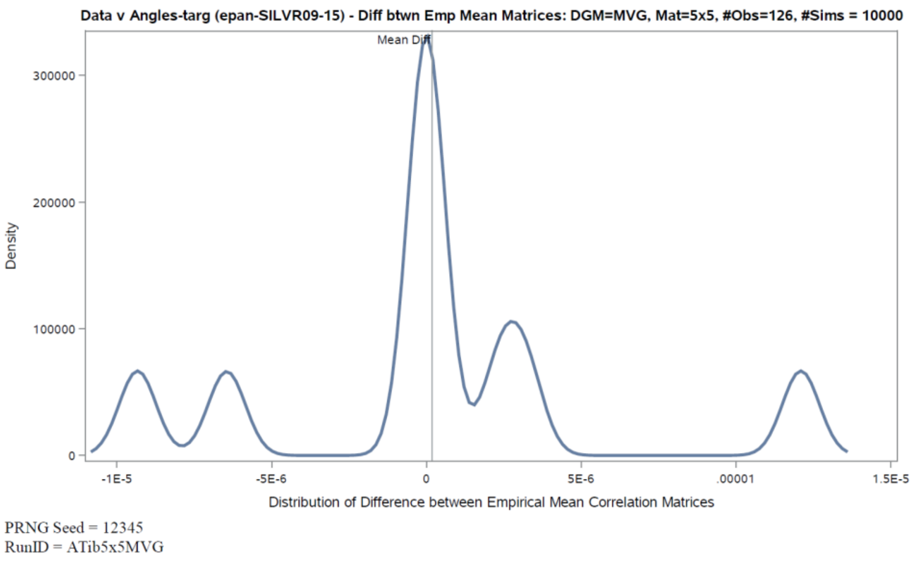


PRNG Seed = 12345
RunID = ATib5x5MVG

Difference Between Mean Empirical Matrices – Both Targeted Scenario

n = 6 **n = 126**

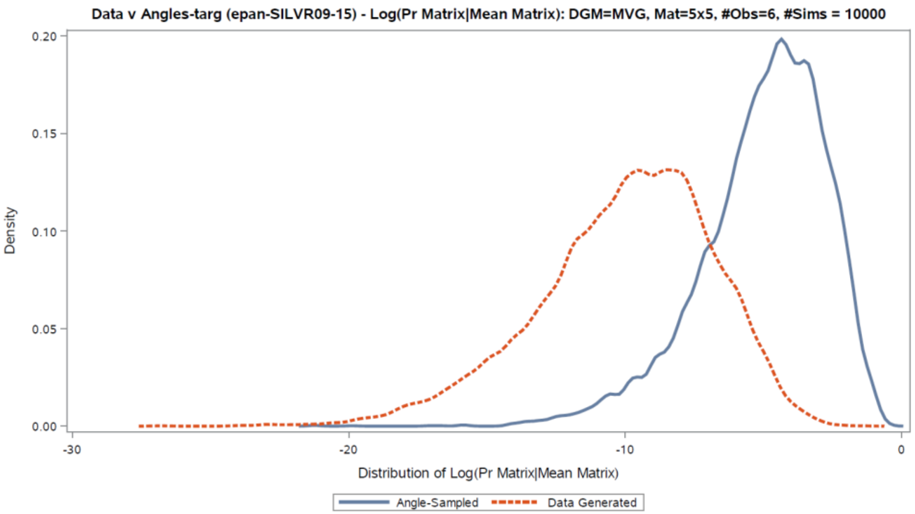




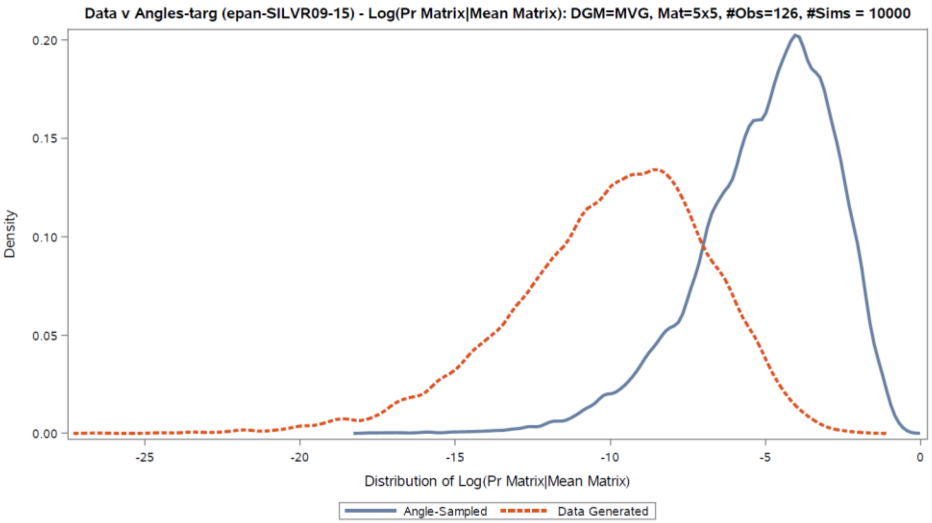
XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case A.TS

LNP – Targeted Scenario v Unconstrained

n = 6 **n = 126**



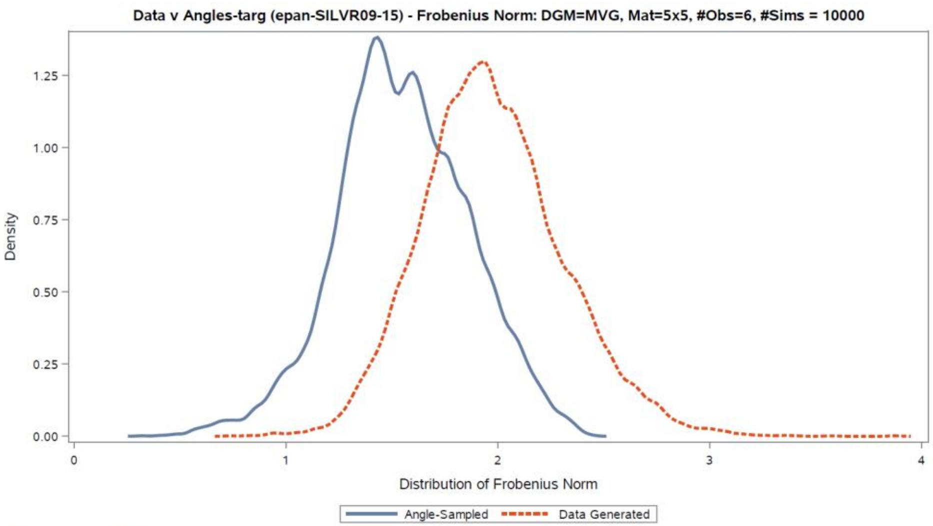
PRNG Seed = 12345
RunID = ATis5x5MVG



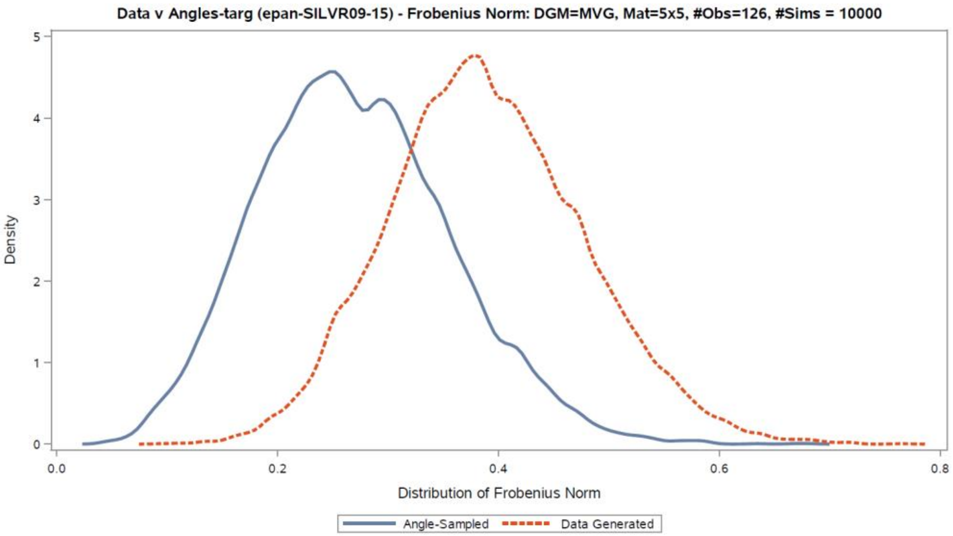
PRNG Seed = 12345
RunID = ATib5x5MVG

Euclidian/Frobenius Norm – Targeted Scenario v Unconstrained

n = 6 | **n = 126**



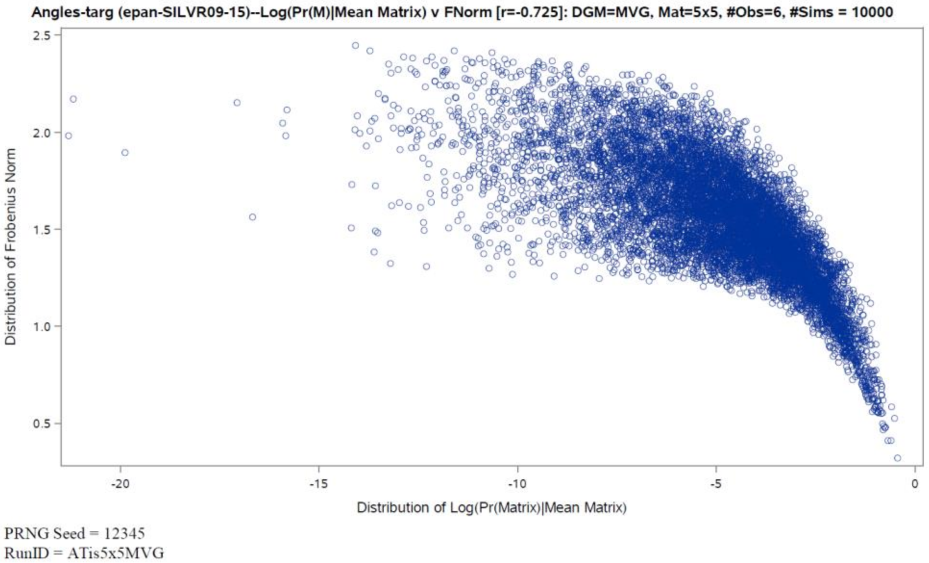
PRNG Seed = 12345
RunID = ATis5x5MVG

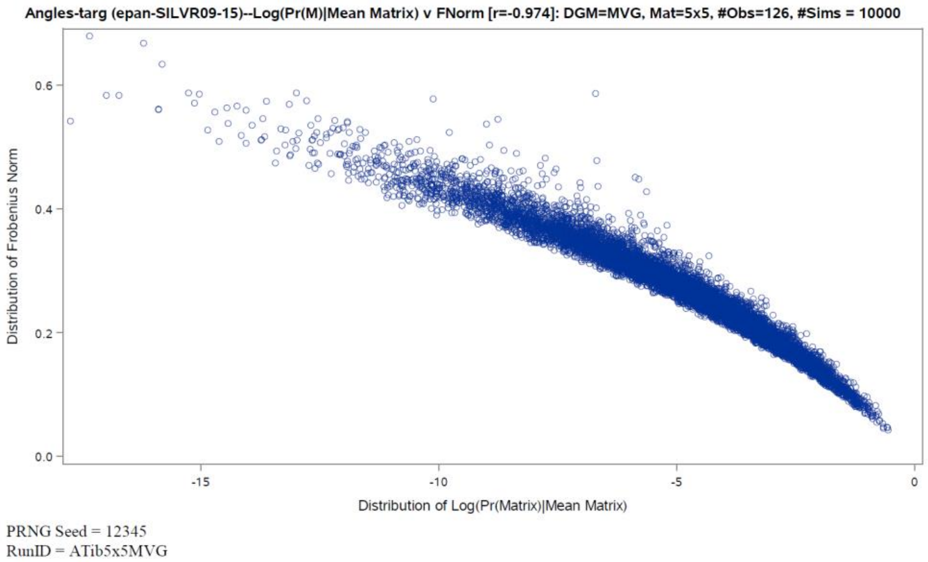


PRNG Seed = 12345
RunID = ATib5x5MVG

LNP v Euclidian/Frobenius Norm – Both Targeted Scenario

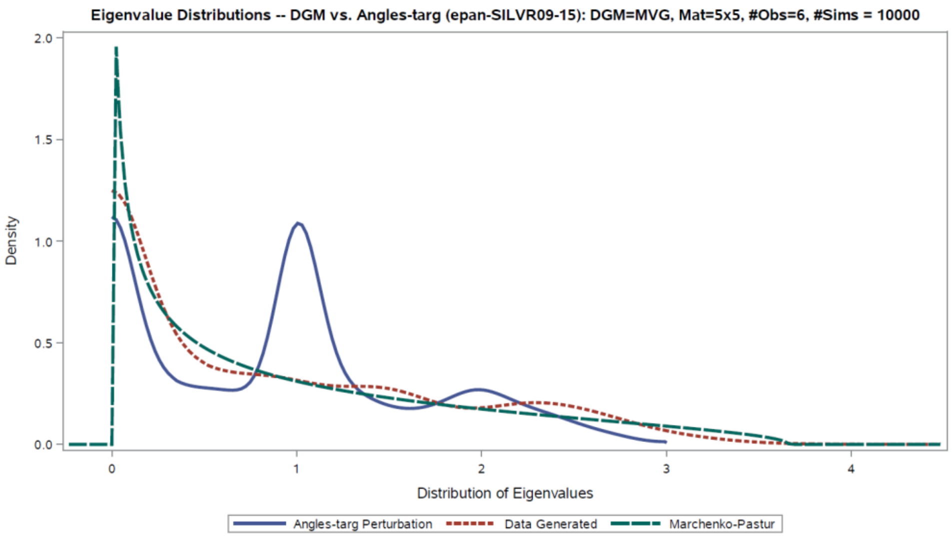
n = 6 **n = 126**



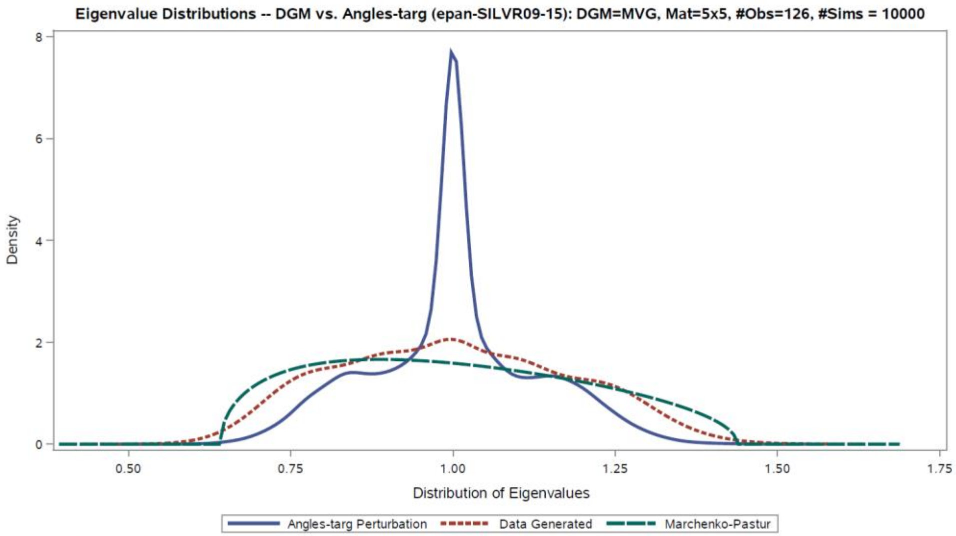


Spectral Distributions – Targeted Scenario v Unconstrained

n = 6 | **n = 126**



PRNG Seed = 12345
RunID = ATis5x5MVG

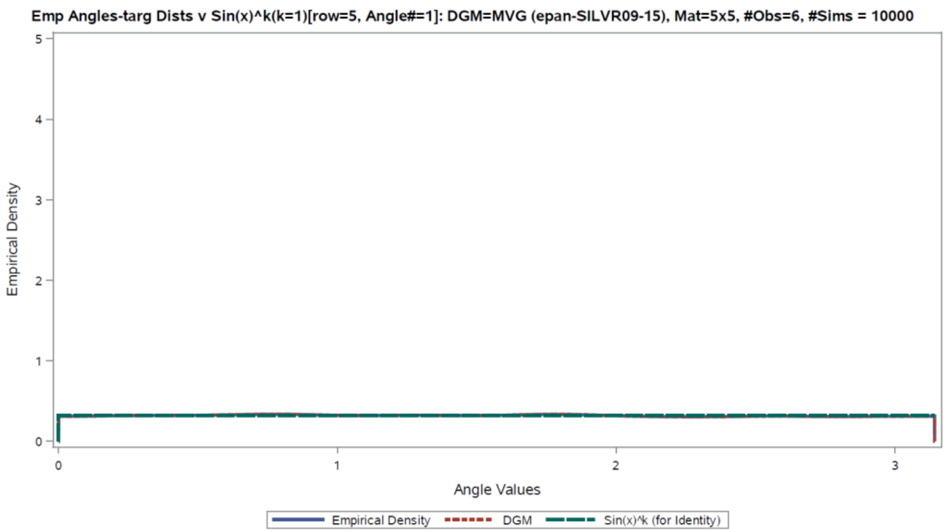


PRNG Seed = 12345
RunID = ATib5x5MVG

XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case A.TS

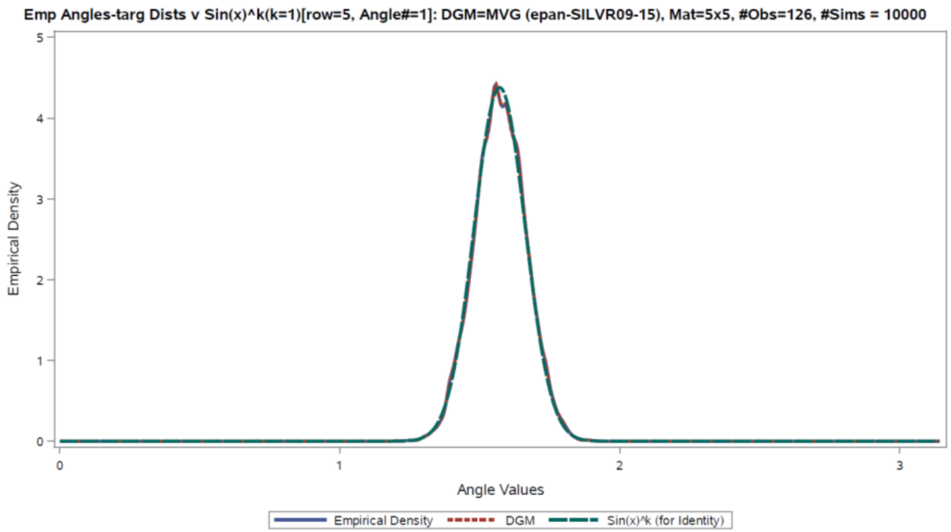
Angle Distributions – Targeted Scenario v Unconstrained

n = 6



PRNG Seed = 12345
RunID = ATis5x5MVG

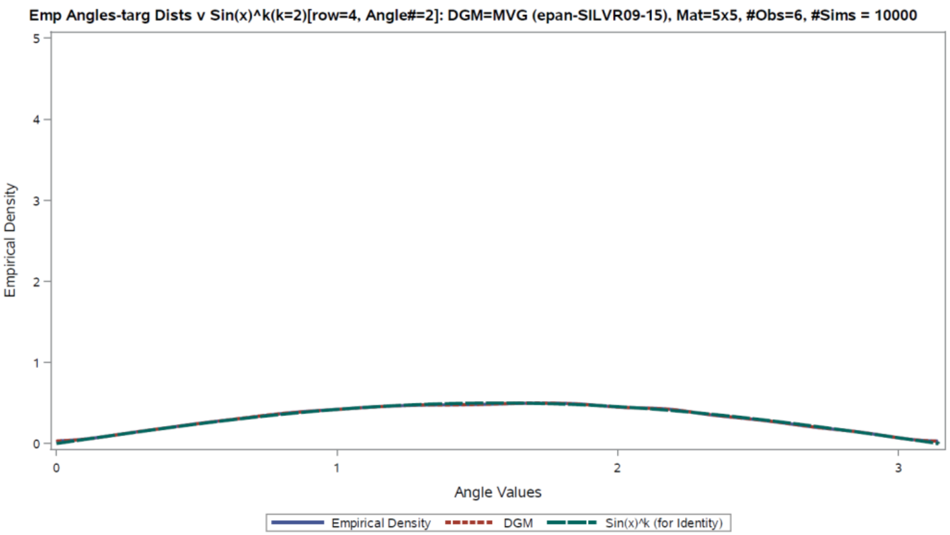
n = 126



PRNG Seed = 12345
RunID = ATib5x5MVG

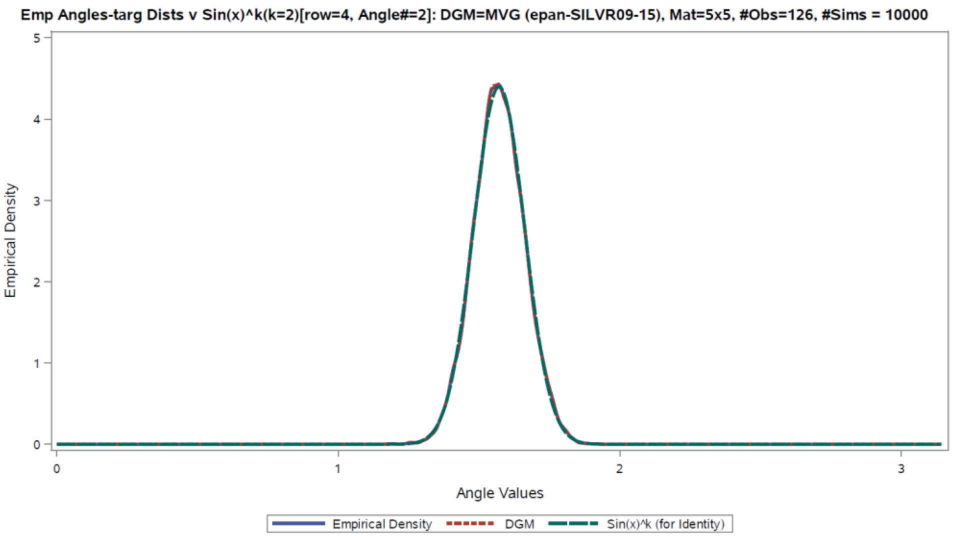
Angle Distributions – Targeted Scenario v Unconstrained

n = 6



PRNG Seed = 12345
RunID = ATis5x5MVG

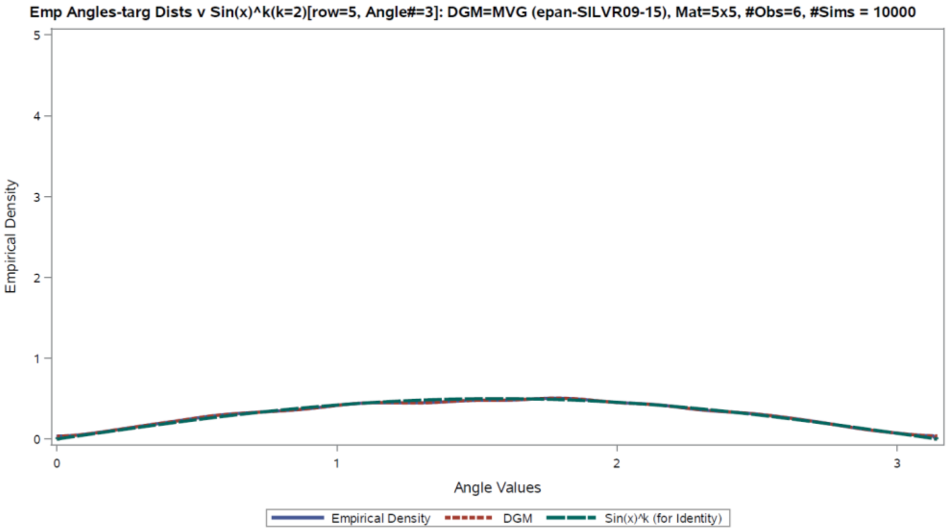
n = 126



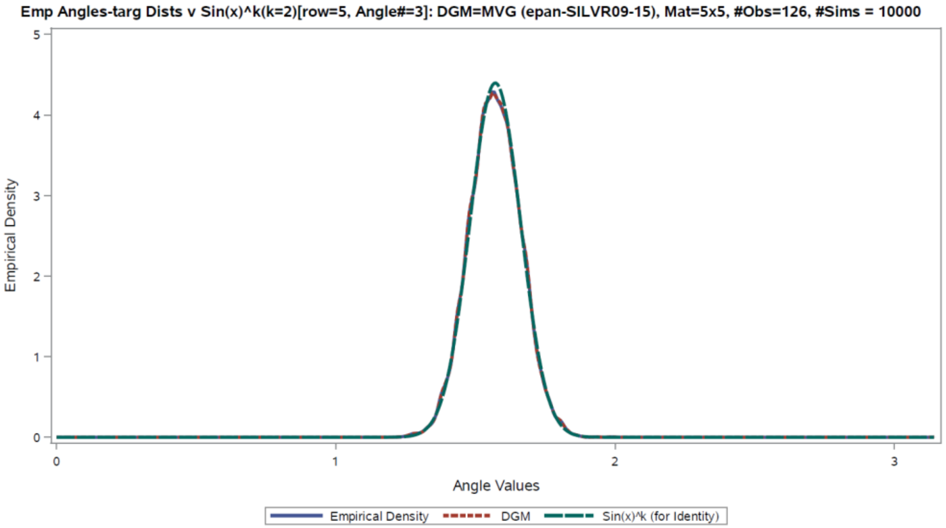
PRNG Seed = 12345
RunID = ATib5x5MVG

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



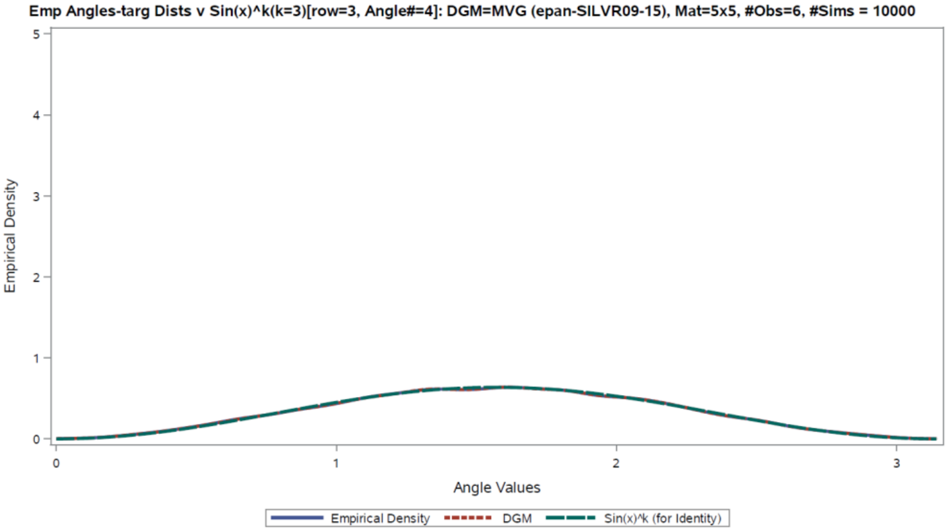
PRNG Seed = 12345
RunID = ATis5x5MVG



PRNG Seed = 12345
RunID = ATib5x5MVG

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



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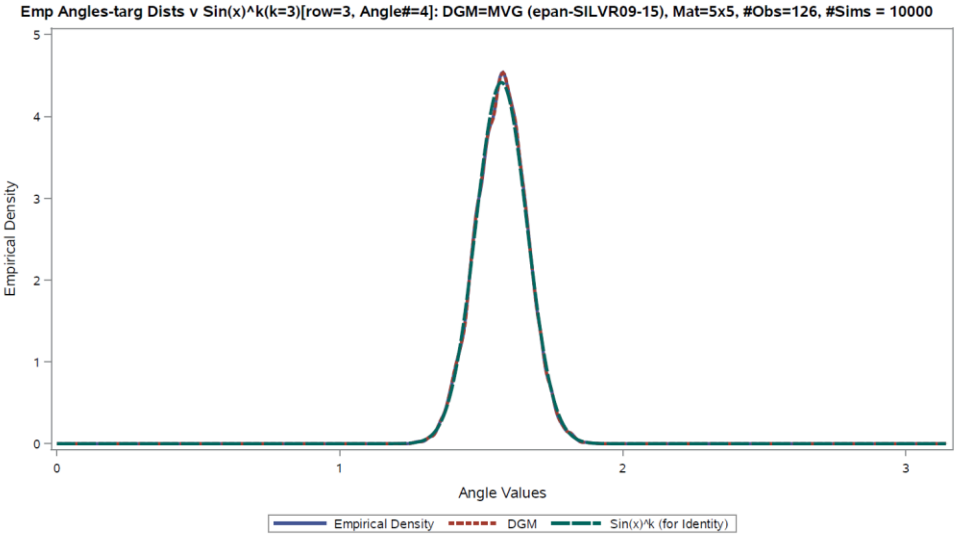
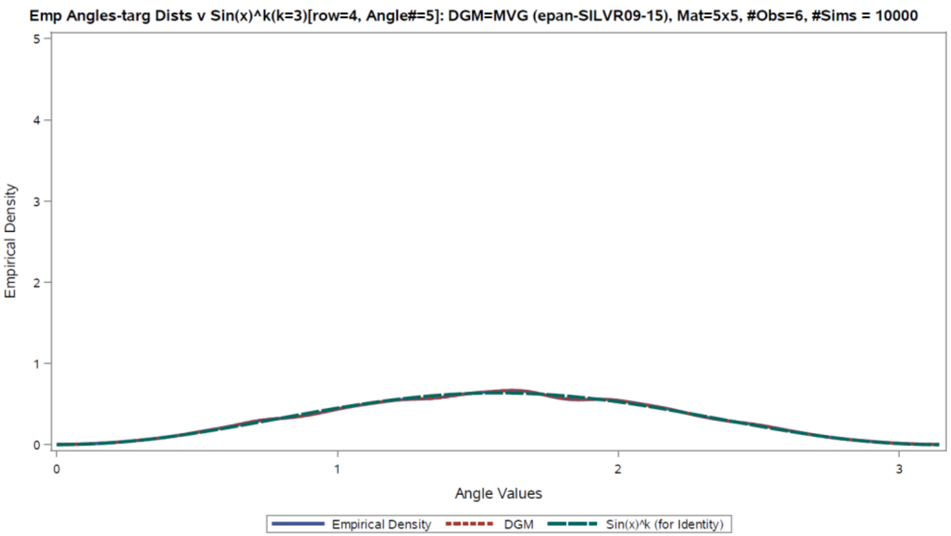


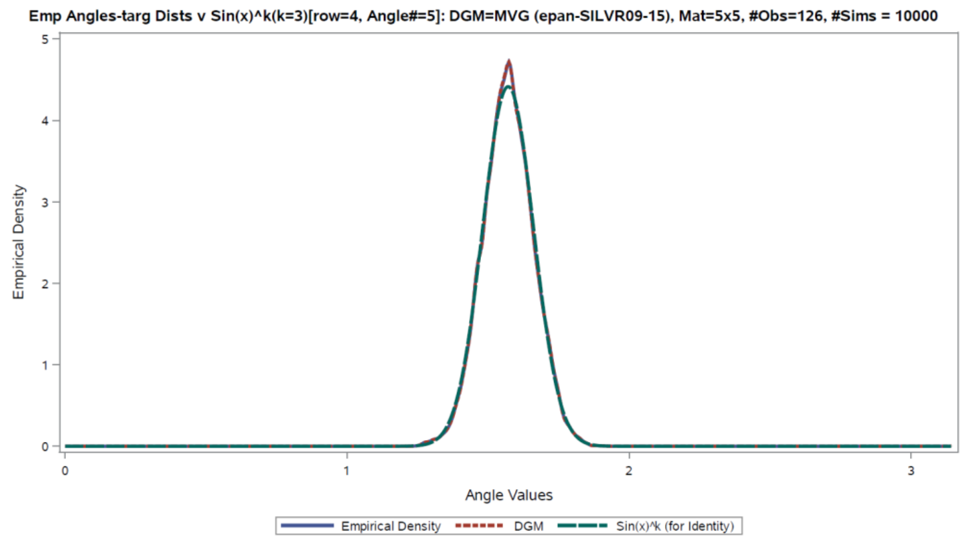
PRNG Seed = 12345
RunID = ATib5x5MVG

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



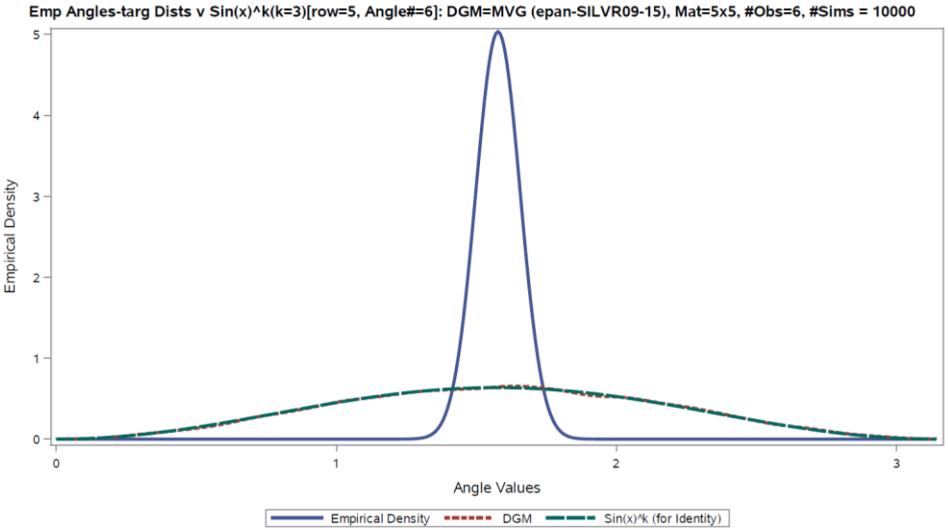
PRNG Seed = 12345
RunID = ATis5x5MVG



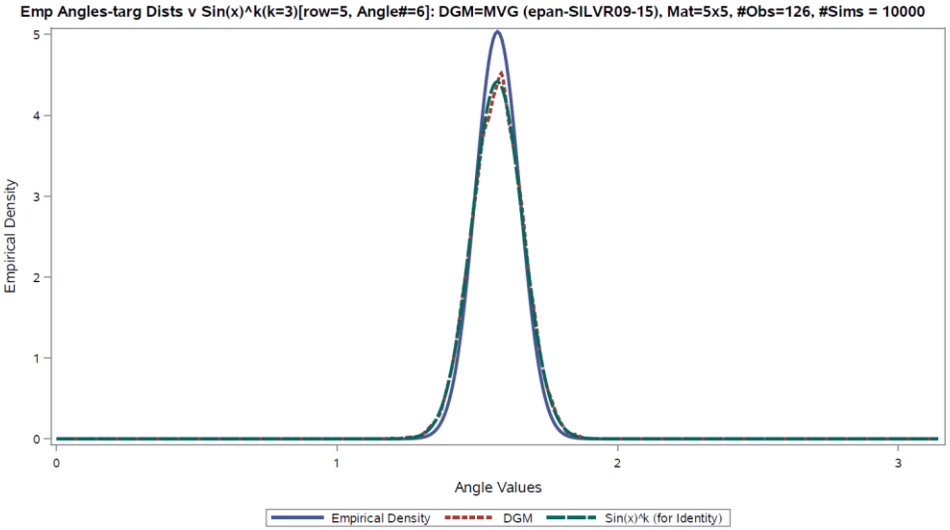
PRNG Seed = 12345
RunID = ATib5x5MVG

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



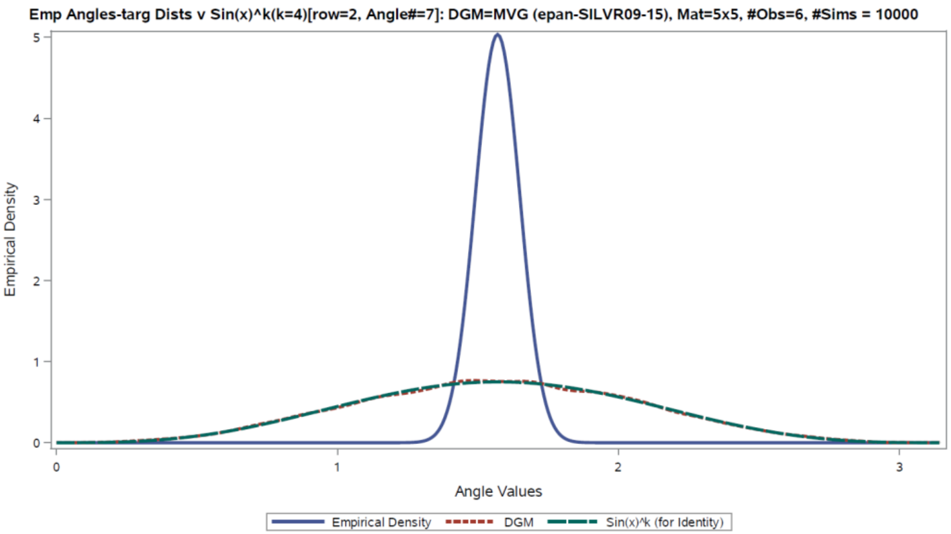
PRNG Seed = 12345
RunID = ATis5x5MVG



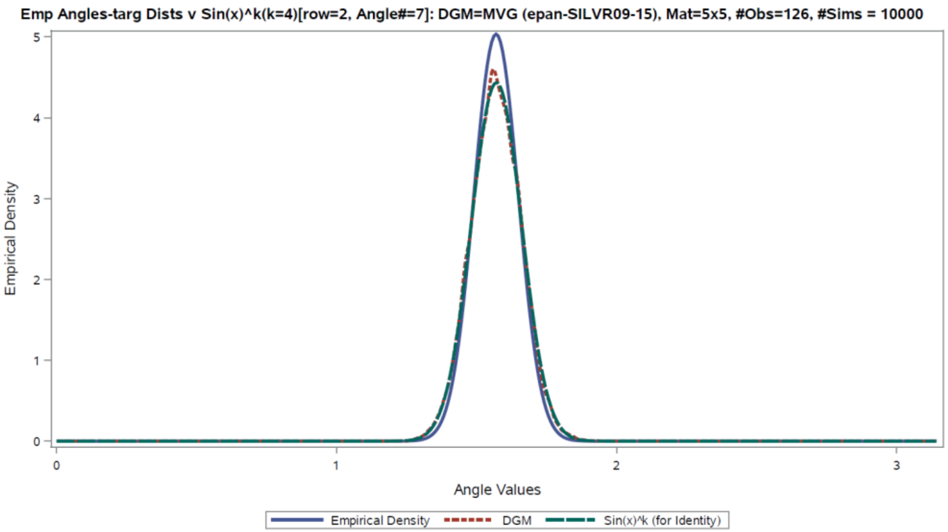
PRNG Seed = 12345
RunID = ATib5x5MVG

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



PRNG Seed = 12345
RunID = ATis5x5MVG

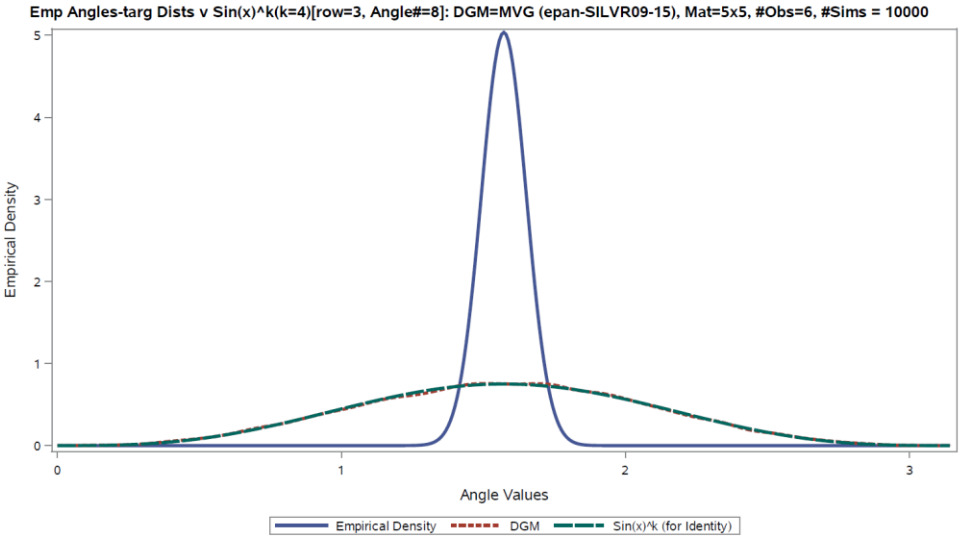


PRNG Seed = 12345
RunID = ATib5x5MVG

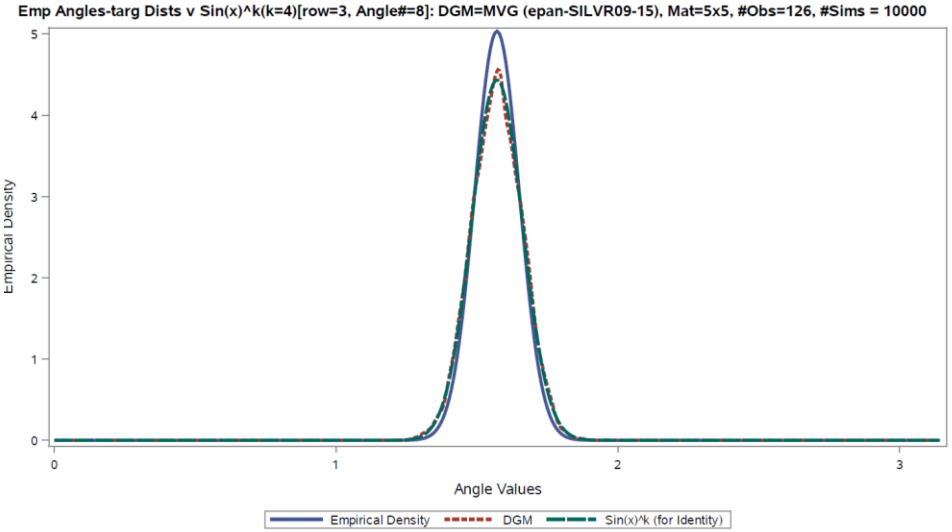
XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case A.TS

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



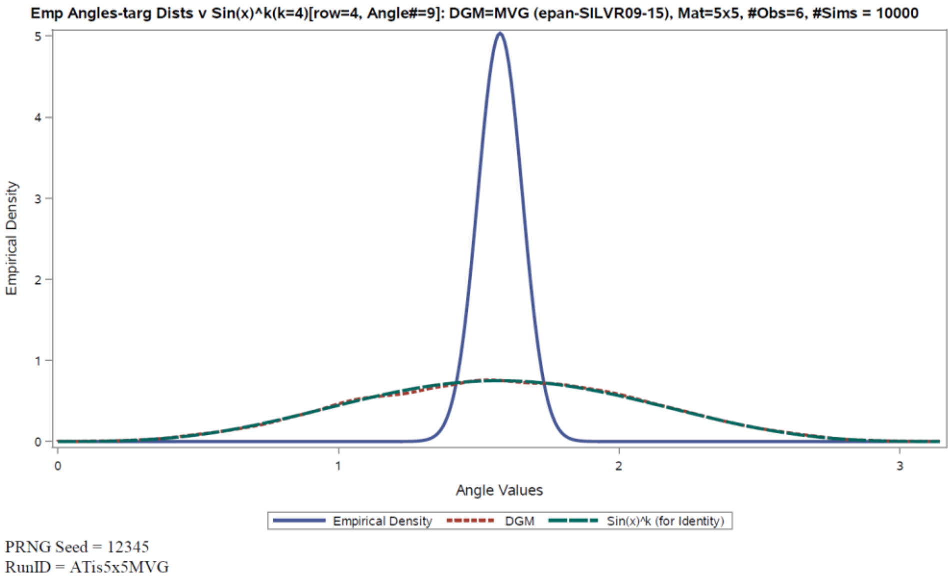
PRNG Seed = 12345
RunID = ATis5x5MVG

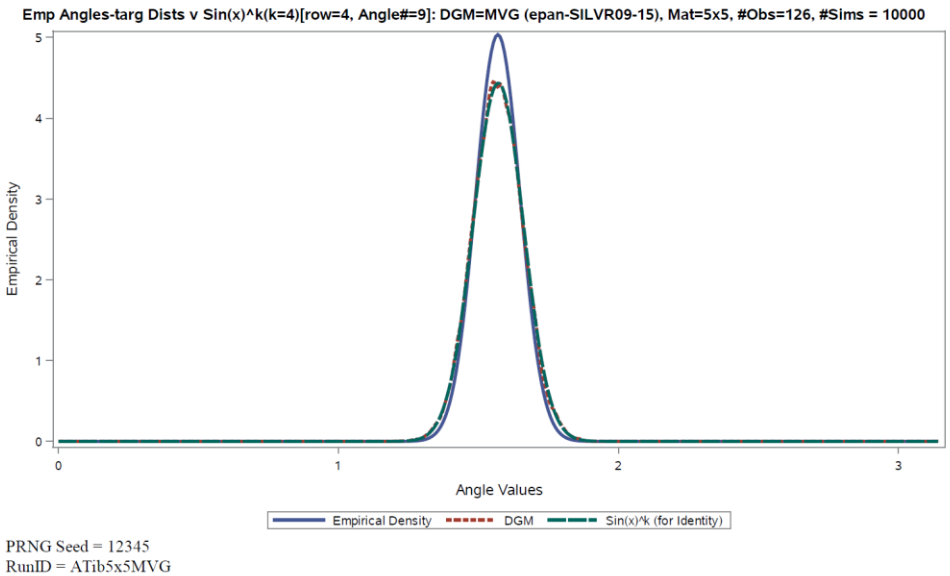


PRNG Seed = 12345
RunID = ATib5x5MVG

Angle Distributions – Targeted Scenario v Unconstrained

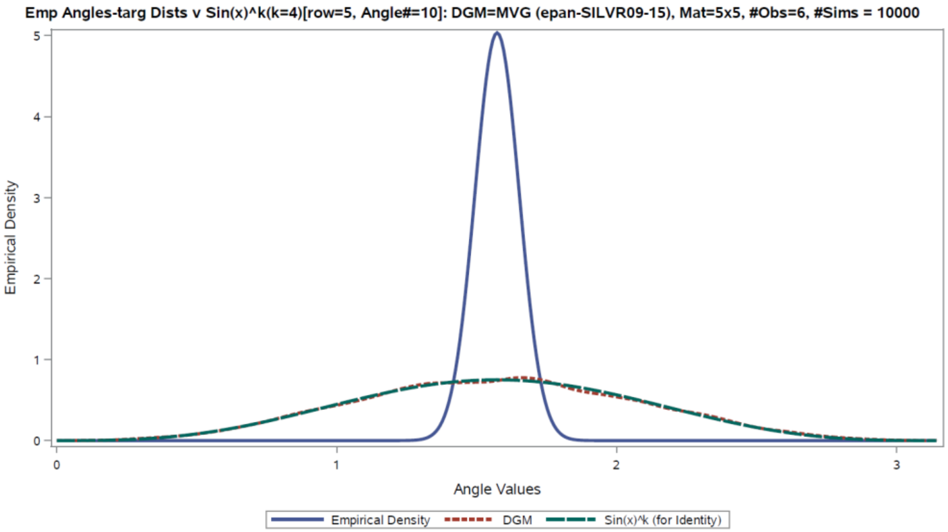
n = 6 **n = 126**



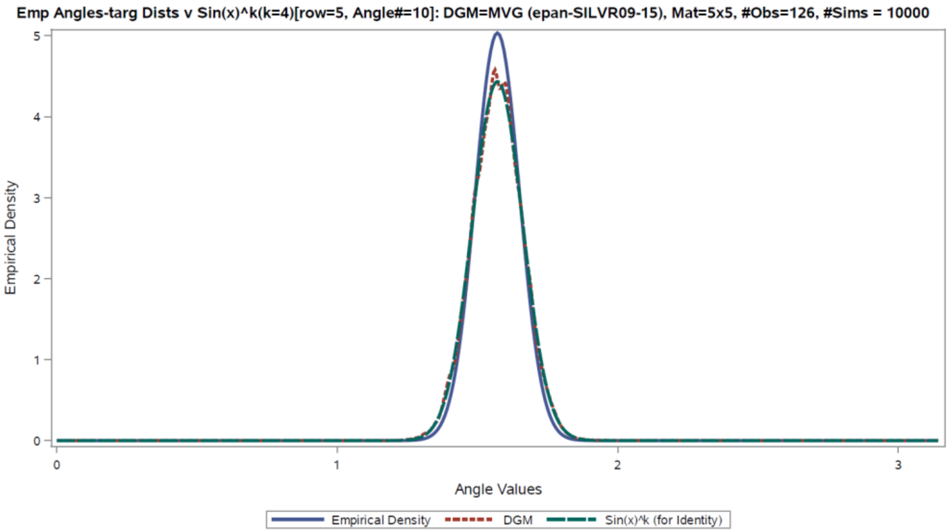


Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



PRNG Seed = 12345
RunID = ATis5x5MVG



PRNG Seed = 12345
RunID = ATib5x5MVG

Correlation Matrices to CDF Matrices – Both Targeted Scenario

Correlation Matrices **n=6** **n=126**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.50	0.50	0.50	0.50		0.503	0.503	0.503	0.503	0.503
0	0.1	0.2	-0.1	-0.1		0.50	0.50	0.50	0.50	0.50		0.499	0.499	0.499	0.499	0.499
0	0.1	0.2	-0.1	-0.1		0.50	0.50	0.50	0.50	0.50		0.501	0.501	0.501	0.501	0.501
0	0.1	0.2	-0.1	-0.1		0.50	0.50	0.50	0.50	0.50		0.501	0.501	0.501	0.501	0.501
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.50	0.44	0.40	0.57	0.38		0.496	0.156	0.032	0.896	0.015
0	0.1	0.2	-0.1	0.2		0.49	0.44	0.39	0.57	0.37		0.500	0.151	0.029	0.892	0.015
0	0.1	0.2	-0.1	0.2		0.50	0.50	0.50	0.50	0.50		0.494	0.494	0.494	0.494	0.494
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.50	0.46	0.43	0.56	0.58		0.502	0.176	0.055	0.915	0.953
0	0.1	0.2	-0.1	-0.1		0.50	0.46	0.43	0.56	0.57		0.508	0.186	0.056	0.920	0.956
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.50	0.48	0.47	0.55	0.43		0.496	0.193	0.084	0.939	0.003
1	1	1	1	1												

CDF Matrices to Correlation Matrices – Both Targeted Scenario

CDF Matrices **n=6** **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.00129	0.00129	0.00129	0.00129	0.00129		0.00138	0.00138	0.00138	0.00138	0.00138
0.5	0.4	0.6	0.35	0.9		-0.00062	-0.00062	-0.00062	-0.00062	-0.00062		0.00002	0.00002	0.00002	0.00002	0.00002
0.5	0.4	0.6	0.35	0.9		-0.00438	-0.00438	-0.00438	-0.00438	-0.00438		0.00156	0.00156	0.00156	0.00156	0.00156
0.5	0.4	0.6	0.35	0.9		-0.00175	-0.00175	-0.00175	-0.00175	-0.00175		0.00023	0.00023	0.00023	0.00023	0.00023
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		0.00032	0.16523	-0.15568	-0.15568	-0.49259		-0.00113	0.02137	-0.02326	-0.02326	-0.07492
0.5	0.4	0.6	0.6	0.8		-0.01012	0.14603	-0.15493	-0.15493	-0.49518		-0.00004	0.02195	-0.02142	-0.02142	-0.07445
0.5	0.4	0.6	0.6	0.8		-0.00599	-0.00599	-0.00599	-0.00599	-0.00599		-0.00083	-0.00083	-0.00083	-0.00083	-0.00083
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.00004	0.22686	-0.16786	0.22690	0.55471		0.00026	0.02398	-0.02178	0.02401	0.05183
0.5	0.4	0.6	0.4	0.3		-0.00484	0.19668	-0.20375	0.19891	0.36323		0.00173	0.02512	-0.02207	0.02515	0.04911
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.01793	0.34785	-0.21883	-0.08378	-0.56057		-0.00120	0.02173	-0.02471	-0.01286	-0.14962
						1	1	1	1	1		1	1	1	1	1

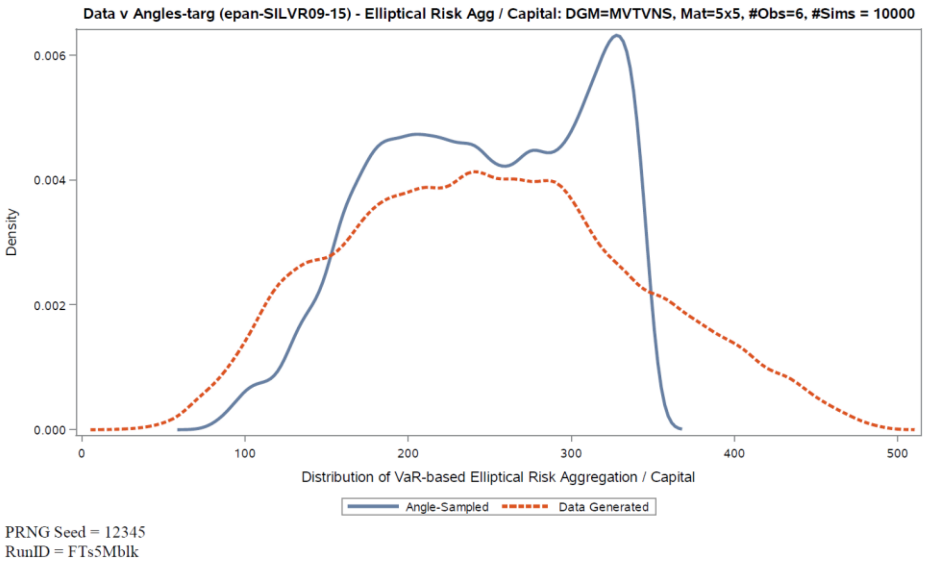
CDF %Shift Matrices to Correlation Matrices – Both Targeted Scenario

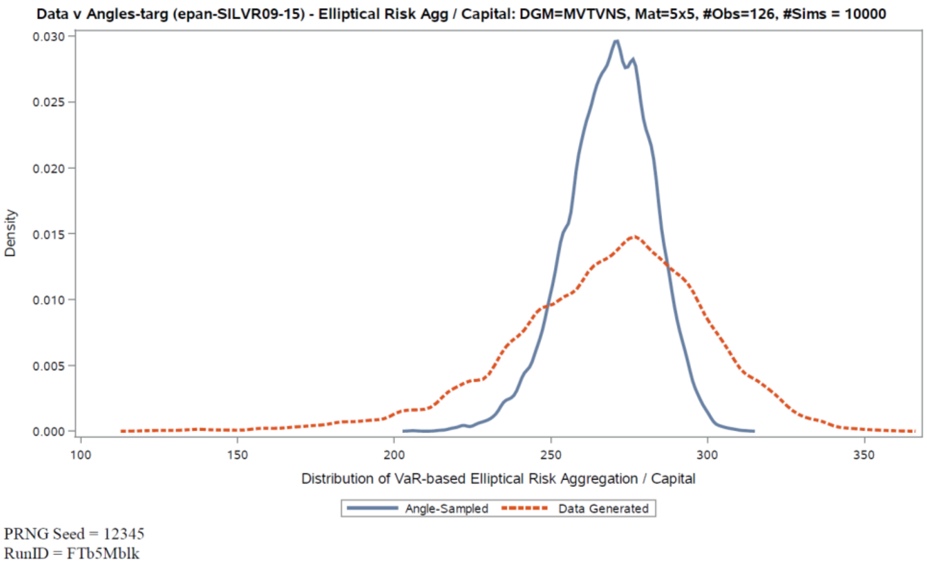
CDF %Shift Matrices **n=6** **n=126**

					lnMatPr	-2.222	-2.241	-6.893	-6.970	-12.755		-2.233	-2.230	-6.938	-6.925	-12.881
					FNorm	0.717	0.584	1.674	1.156	2.241		0.073	0.072	0.195	0.187	0.378
					Rnk_lnMat	1	2	3	4	5		2	1	4	3	5
					Rnk_FNorr	2	1	4	3	5		2	1	4	3	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.0013	0.0013	0.0013	0.0013	0.0013		0.00138	0.00138	0.00138	0.00138	0.00138
20	-20	50	-50	60		-0.0006	-0.0006	-0.0006	-0.0006	-0.0006		0.00002	0.00002	0.00002	0.00002	0.00002
20	-20	50	-50	60		-0.0044	-0.0044	-0.0044	-0.0044	-0.0044		0.00156	0.00156	0.00156	0.00156	0.00156
20	-20	50	-50	60		-0.0017	-0.0017	-0.0017	-0.0017	-0.0017		0.00023	0.00023	0.00023	0.00023	0.00023
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.1698	-0.1509	0.4105	-0.4035	0.5916		0.02242	-0.02225	0.06101	-0.05946	0.09345
20	-20	50	-50	70		0.1517	-0.1495	0.4049	-0.4066	0.5885		0.02188	-0.02150	0.05874	-0.05964	0.09178
20	-20	50	-50	70		-0.0060	-0.0060	-0.0060	-0.0060	-0.0060		-0.00083	-0.00083	-0.00083	-0.00083	-0.00083
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.2325	-0.1663	0.5953	-0.2490	0.8728		0.02348	-0.02251	0.06302	-0.05724	0.12099
20	-20	50	-50	80		0.2028	-0.1972	0.4656	-0.4548	0.6567		0.02469	-0.02254	0.06181	-0.06089	0.11648
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.3405	-0.2326	0.7159	-0.2490	0.8010		0.02112	-0.02547	0.06286	-0.06024	0.15955
						1	1	1	1	1		1	1	1	1	1

Elliptical Capital – Targeted Scenario v Unconstrained

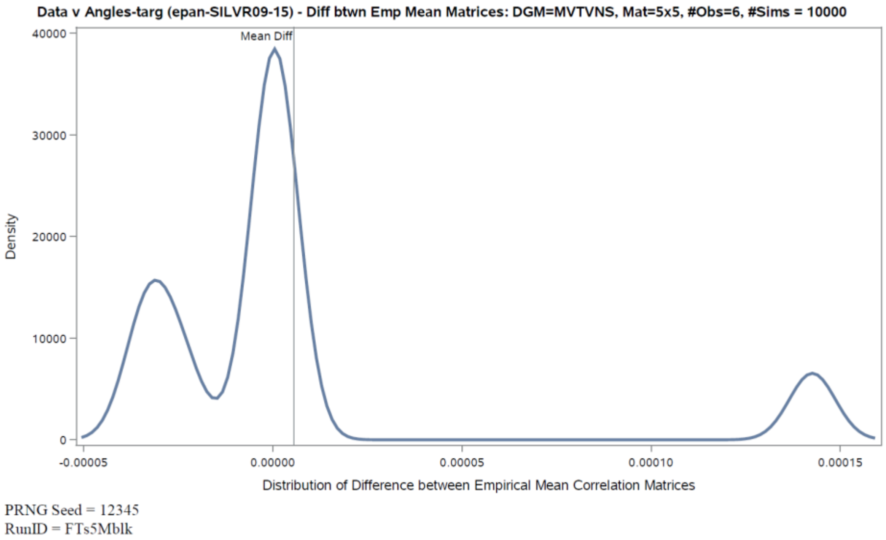
n = 6 **n = 126**

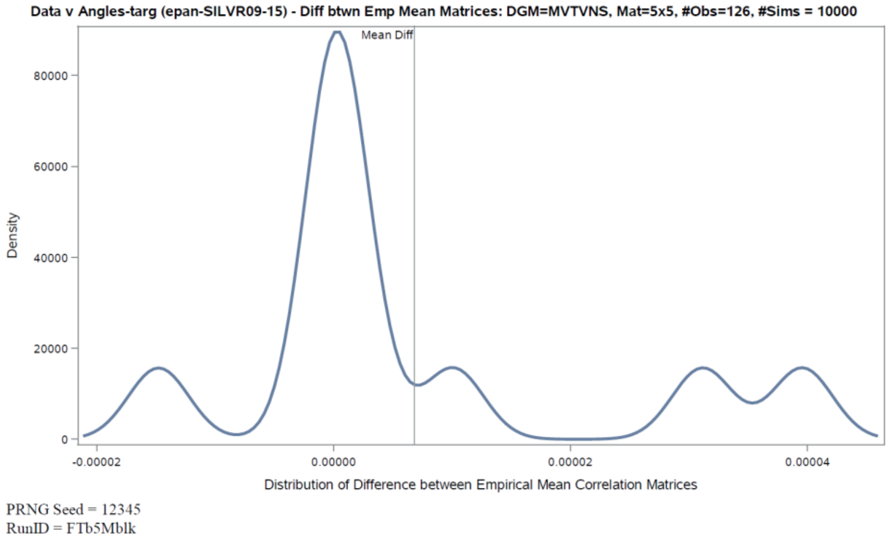




Difference Between Mean Empirical Matrices – Both Targeted Scenario

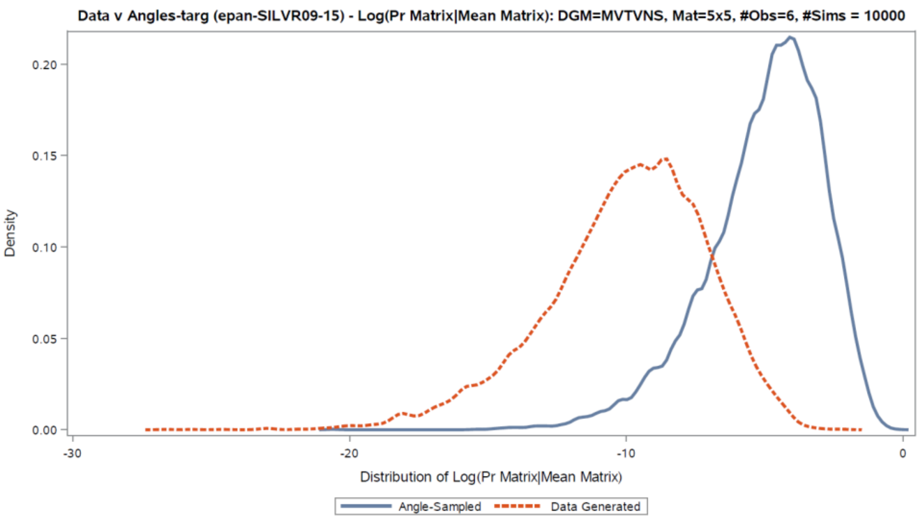
n = 6 **n = 126**





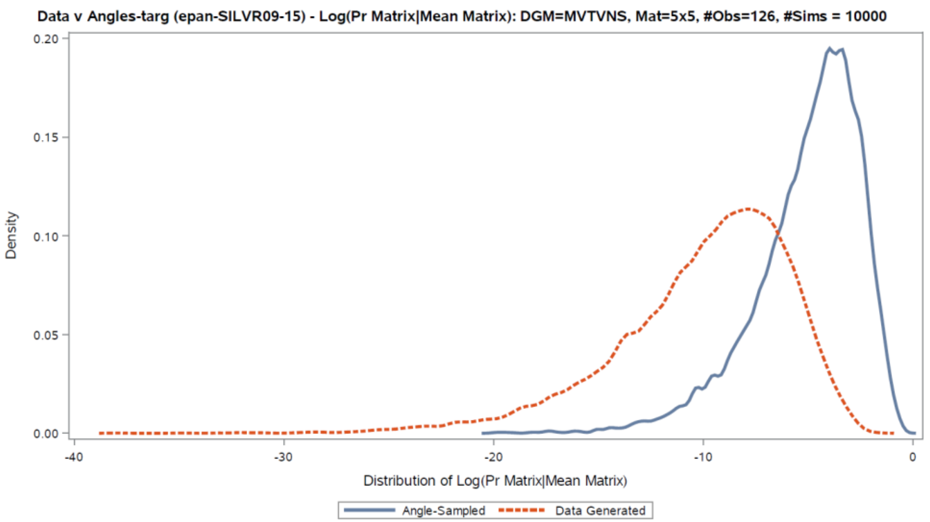
LNP – Targeted Scenario v Unconstrained

n = 6



PRNG Seed = 12345
RunID = FTs5Mblk

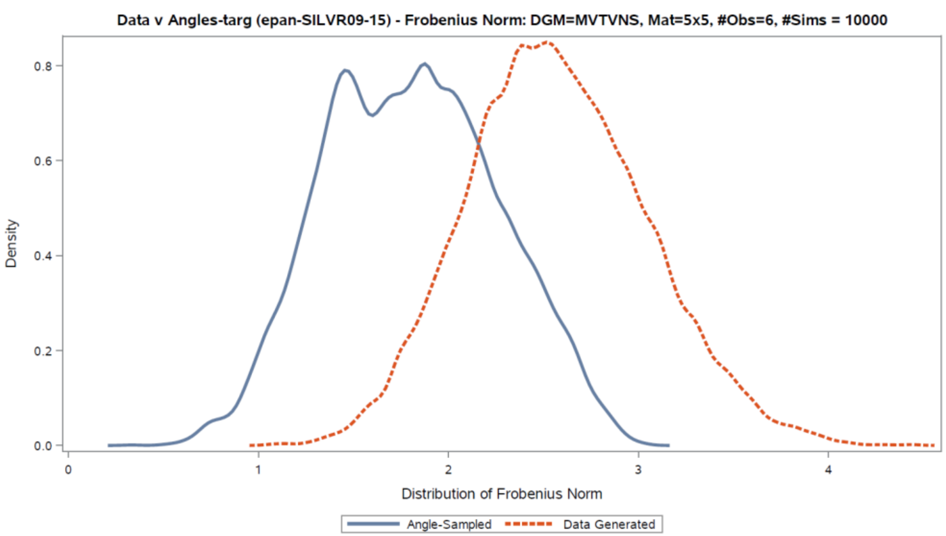
n = 126



PRNG Seed = 12345
RunID = FTb5Mblk

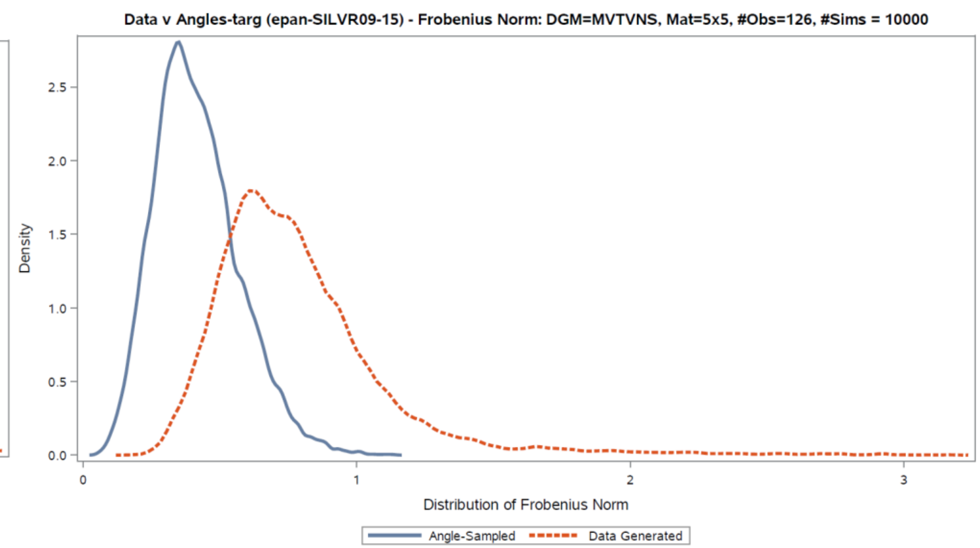
Euclidian/Frobenius Norm – Targeted Scenario v Unconstrained

n = 6



PRNG Seed = 12345
RunID = FTs5Mblk

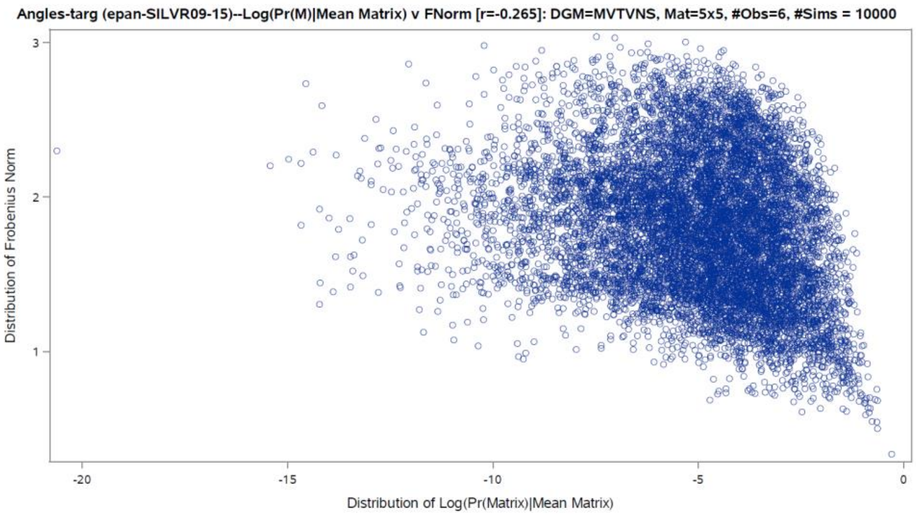
n = 126



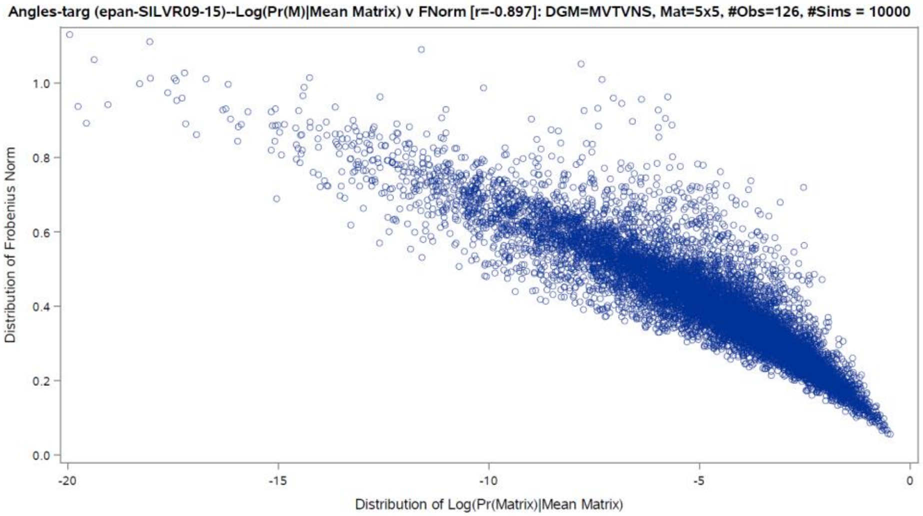
PRNG Seed = 12345
RunID = FTb5Mblk

LNP v Euclidian/Frobenius Norm – Both Targeted Scenario

n = 6 **n = 126**



PRNG Seed = 12345
RunID = FTs5Mblk

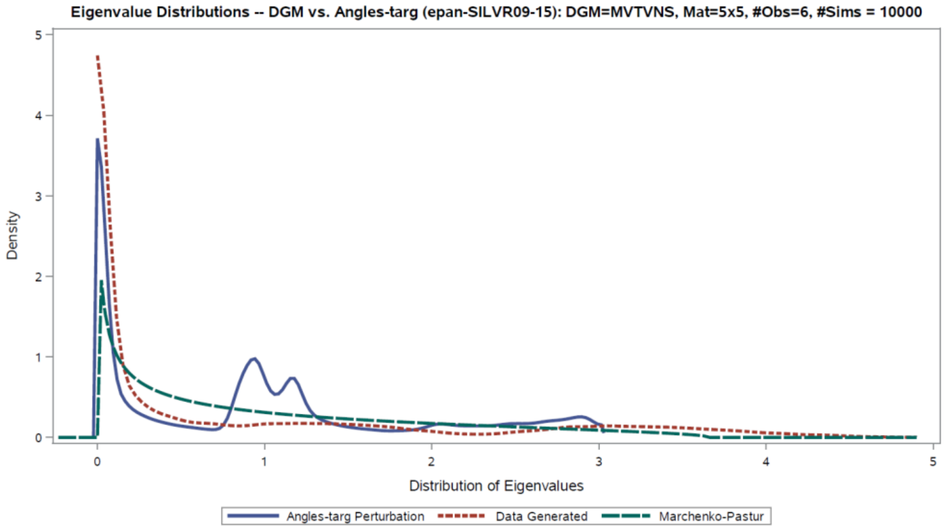


PRNG Seed = 12345
RunID = FTb5Mblk

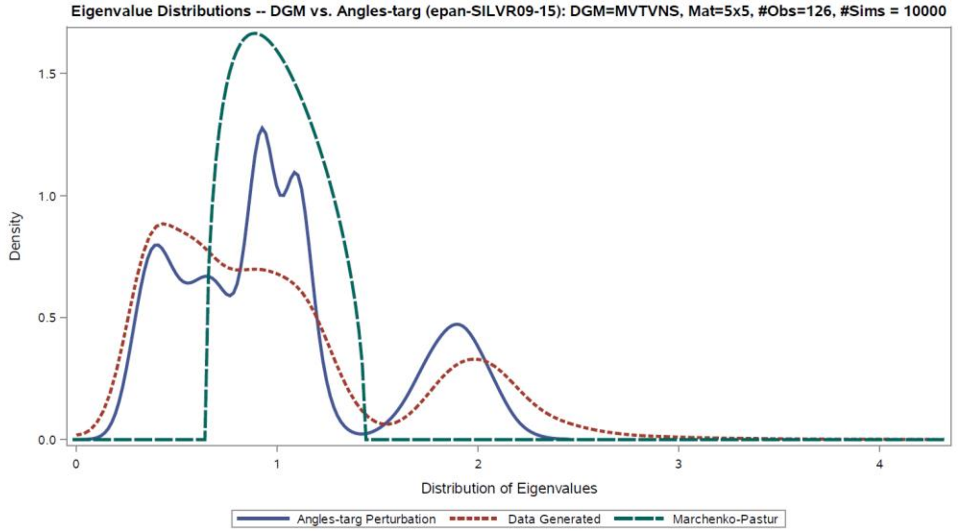
XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case F.TS

Spectral Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



PRNG Seed = 12345
RunID = FTs5Mblk

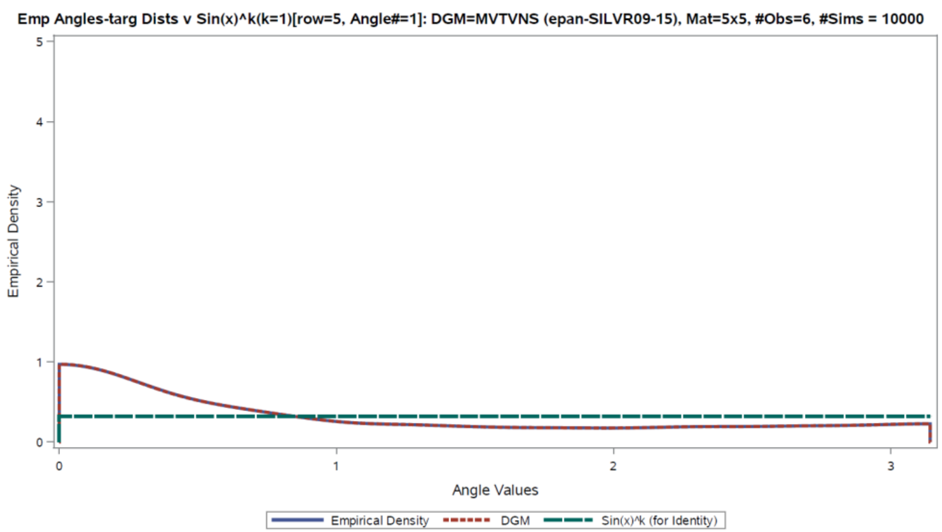


PRNG Seed = 12345
RunID = FTb5Mblk

XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case F.TS

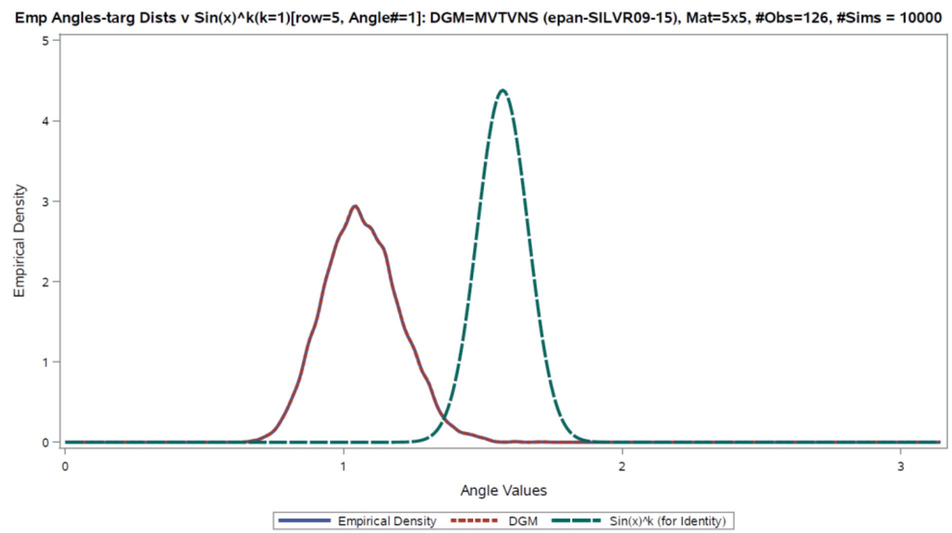
Angle Distributions – Targeted Scenario v Unconstrained

n = 6



PRNG Seed = 12345
RunID = FTs5Mblk

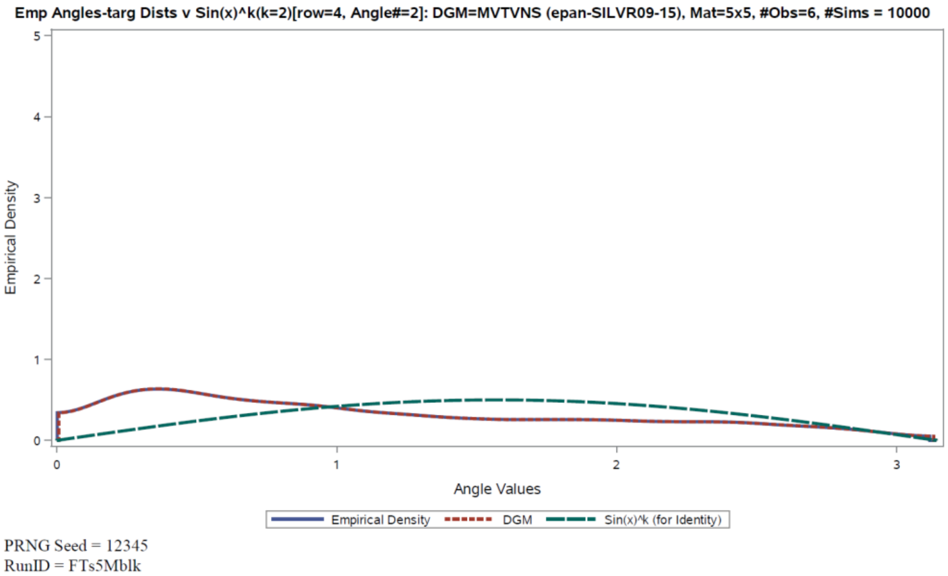
n = 126



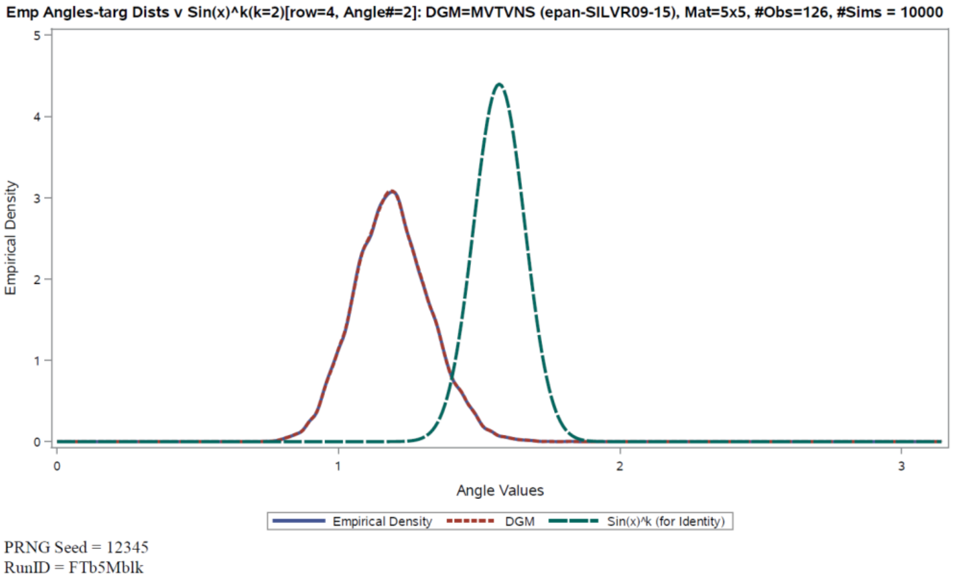
PRNG Seed = 12345
RunID = FTb5Mblk

Angle Distributions – Targeted Scenario v Unconstrained

n = 6

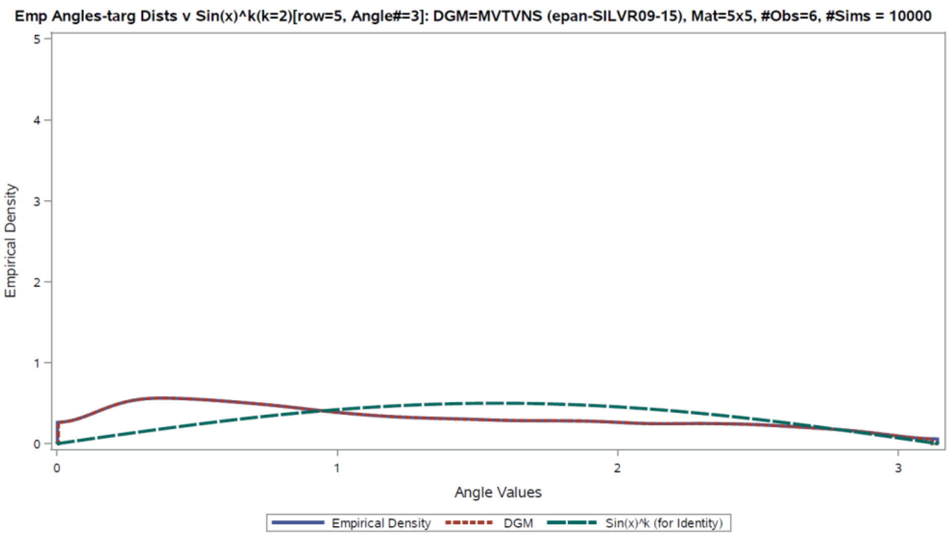


n = 126

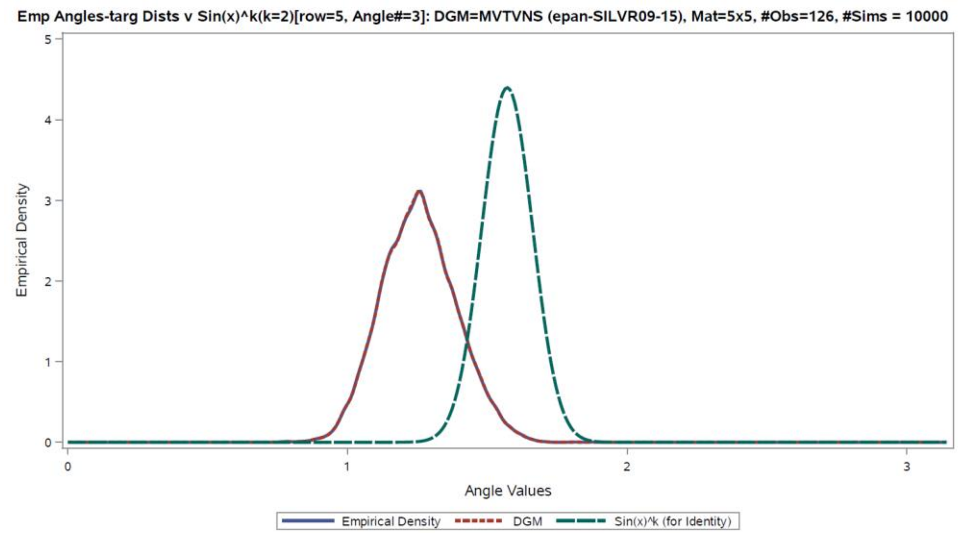


Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



PRNG Seed = 12345
RunID = FTs5Mblk

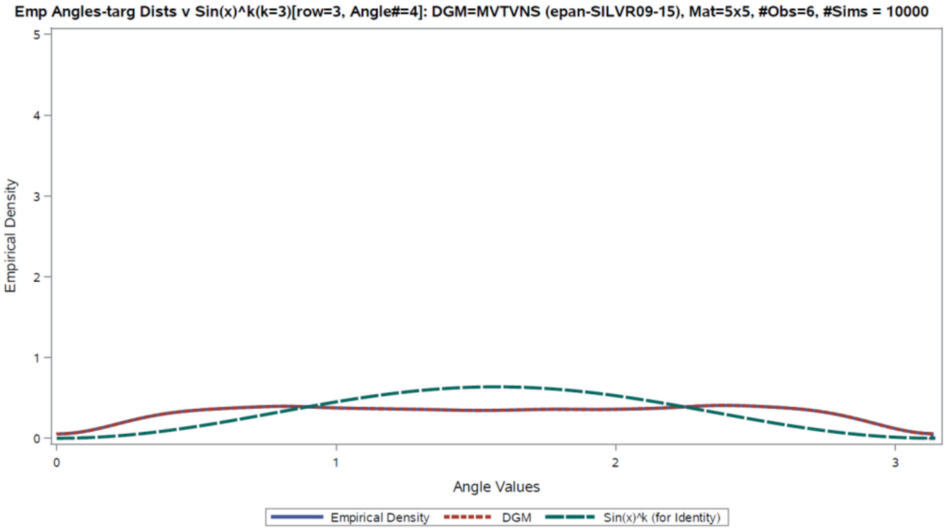


PRNG Seed = 12345
RunID = FTb5Mblk

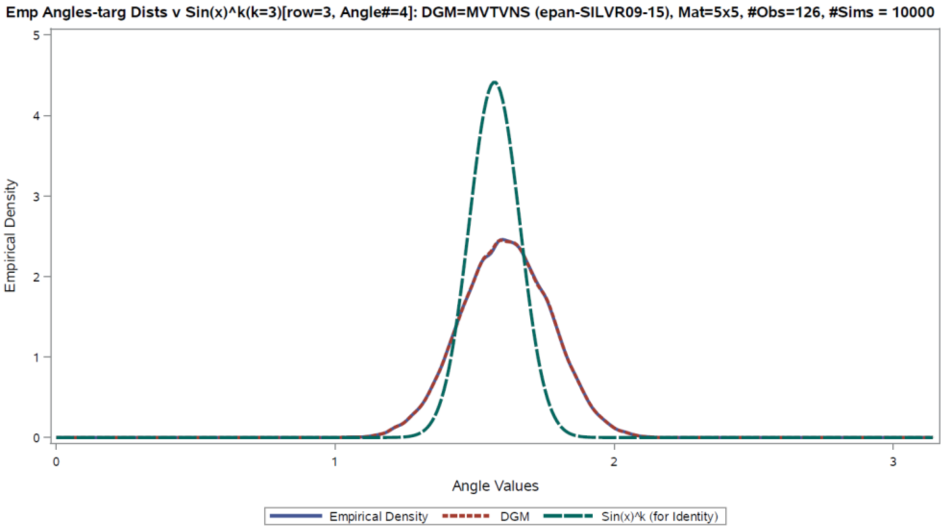
XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case F.TS

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



PRNG Seed = 12345
RunID = FTs5Mblk

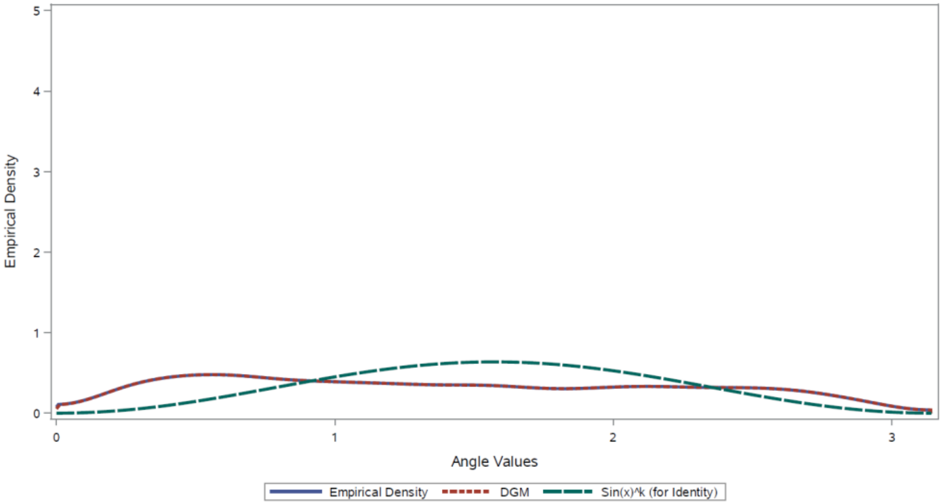


PRNG Seed = 12345
RunID = FTb5Mblk

Angle Distributions – Targeted Scenario v Unconstrained

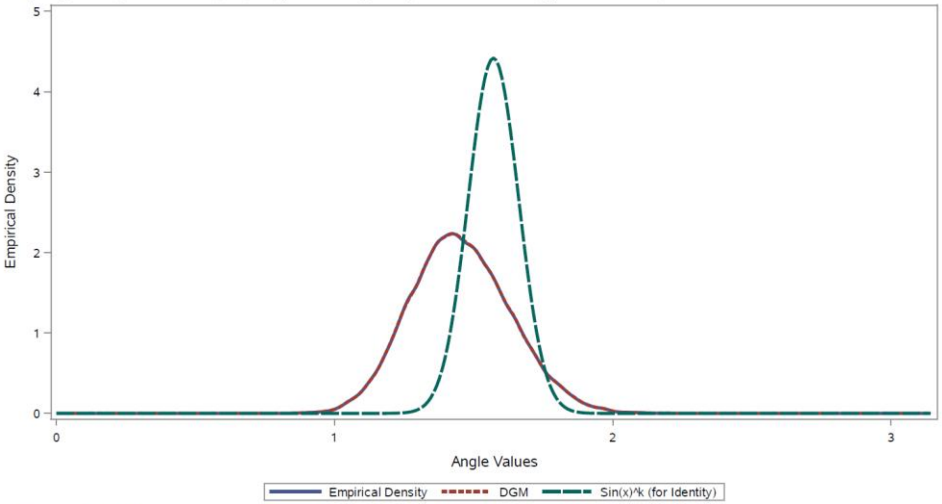
n = 6 **n = 126**

Emp Angles-targ Dists v Sin(x)^k(k=3)[row=4, Angle#=5]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=6, #Sims = 10000



PRNG Seed = 12345
RunID = FTs5Mblk

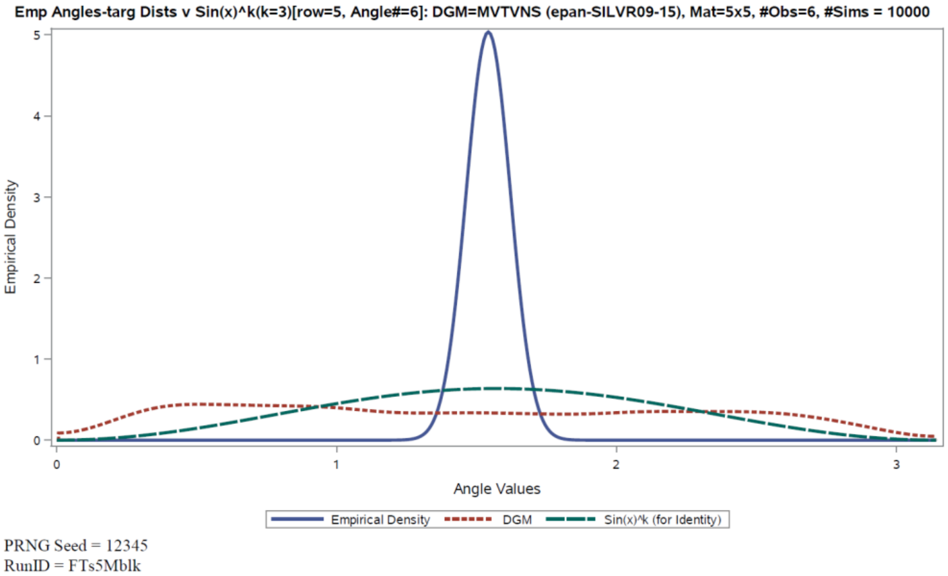
Emp Angles-targ Dists v Sin(x)^k(k=3)[row=4, Angle#=5]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000

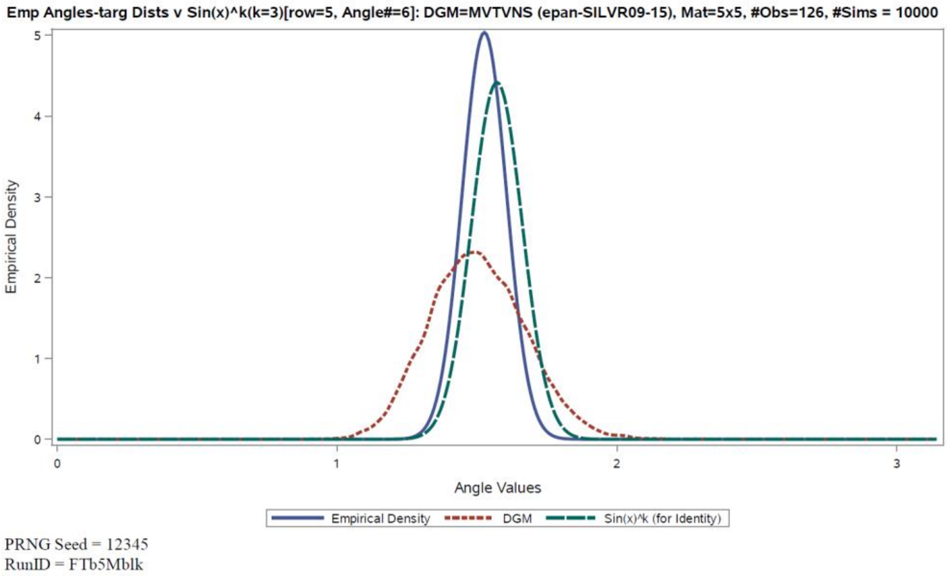


PRNG Seed = 12345
RunID = FTb5Mblk

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**

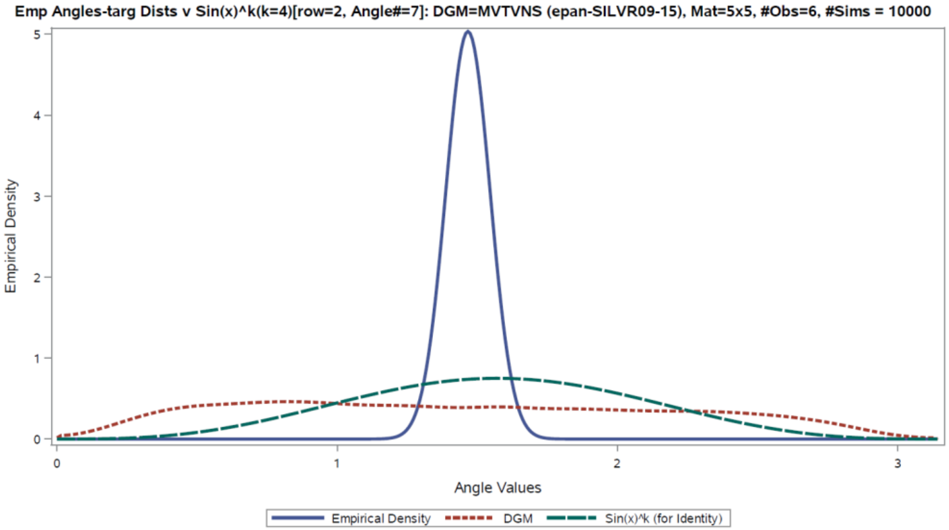




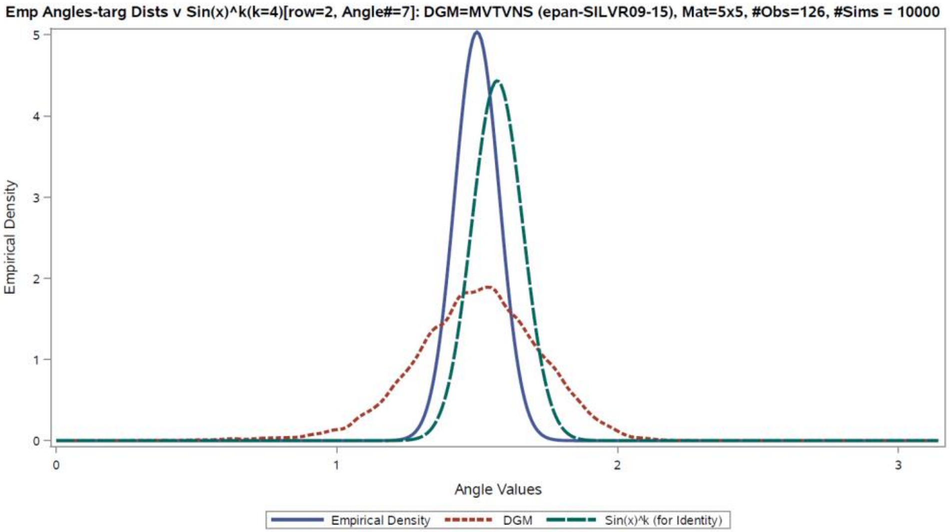
XI. APPENDIX 2: Empirical Results of NAbC, Full Matrix – Case F.TS

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



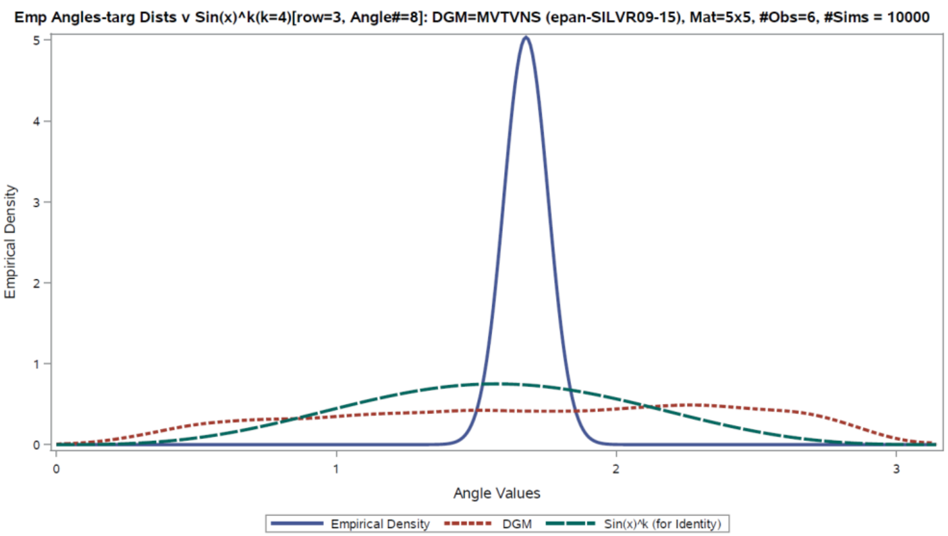
PRNG Seed = 12345
RunID = FTs5Mblk



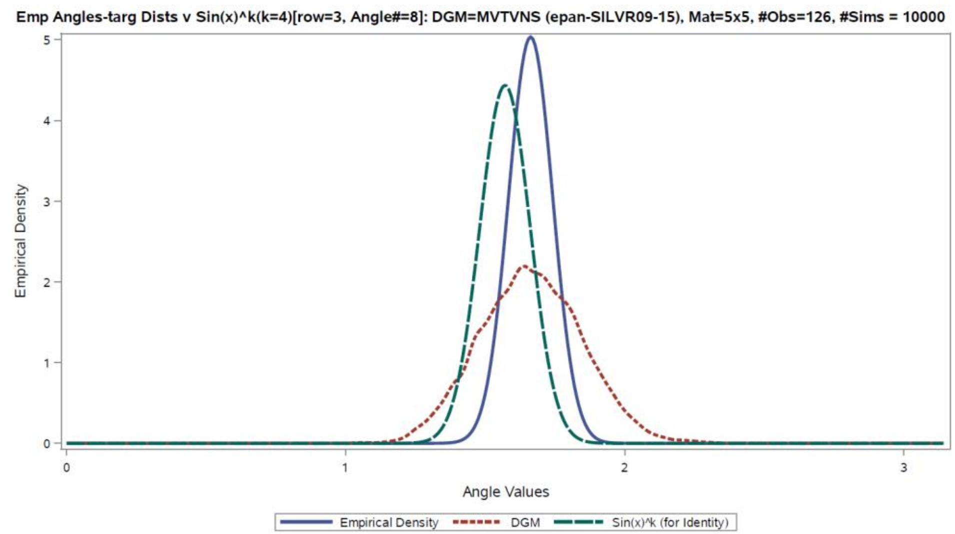
PRNG Seed = 12345
RunID = FTb5Mblk

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 **n = 126**



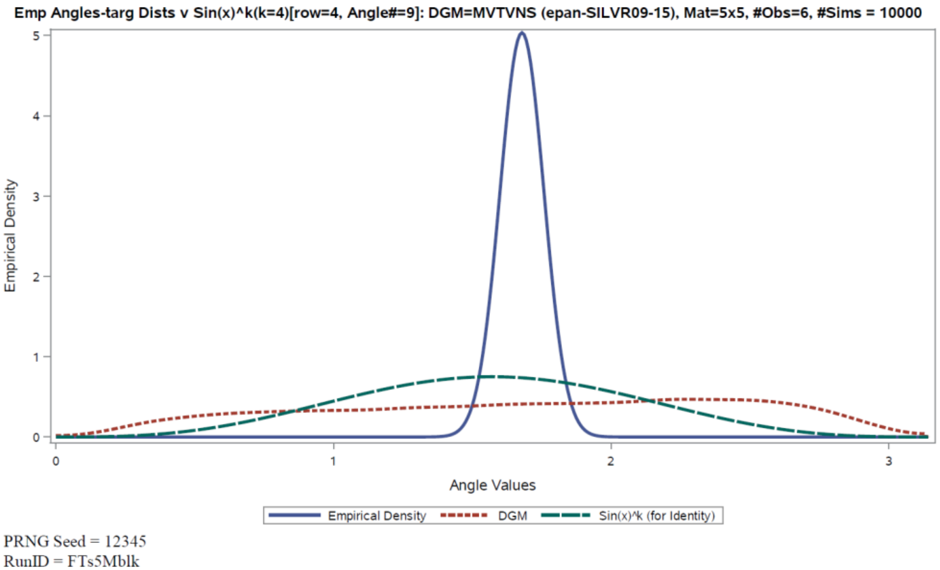
PRNG Seed = 12345
RunID = FTs5Mblk

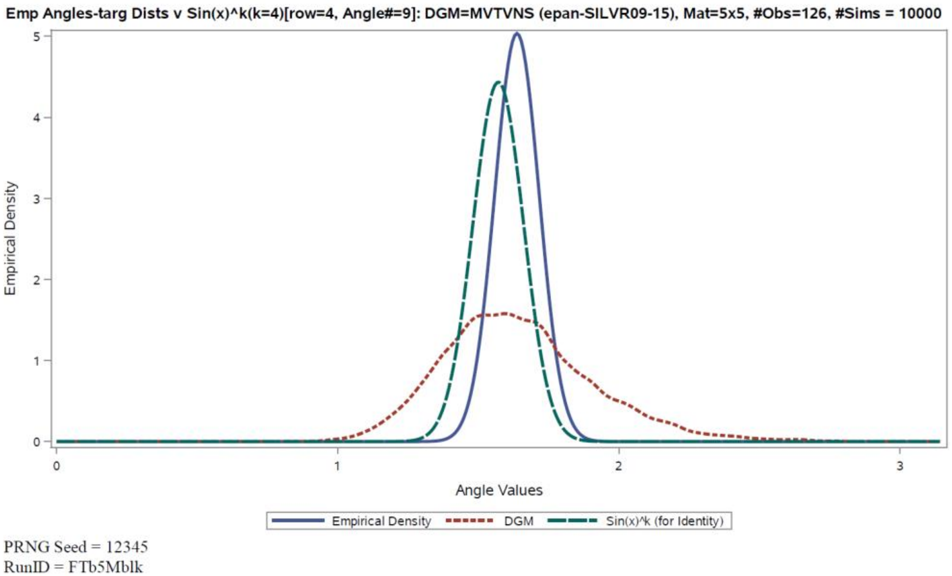


PRNG Seed = 12345
RunID = FTb5Mblk

Angle Distributions – Targeted Scenario v Unconstrained

n = 6 n = 126

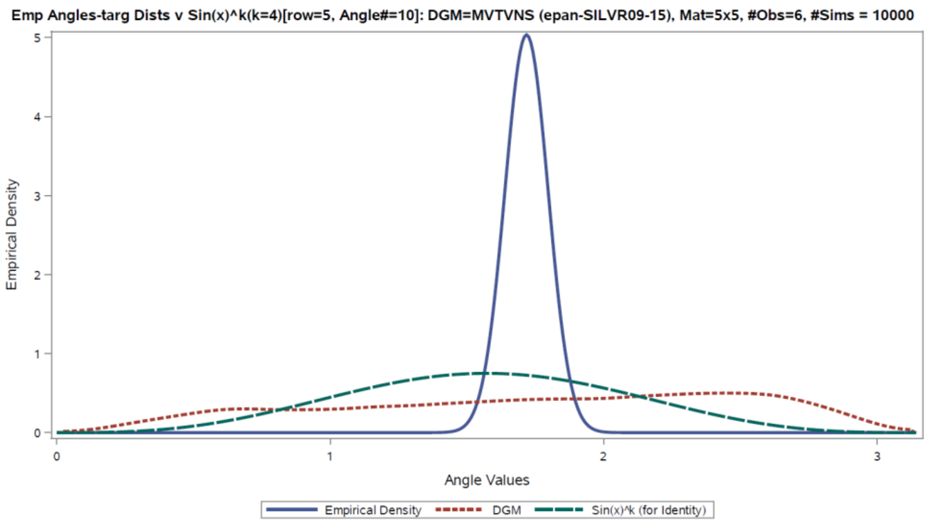




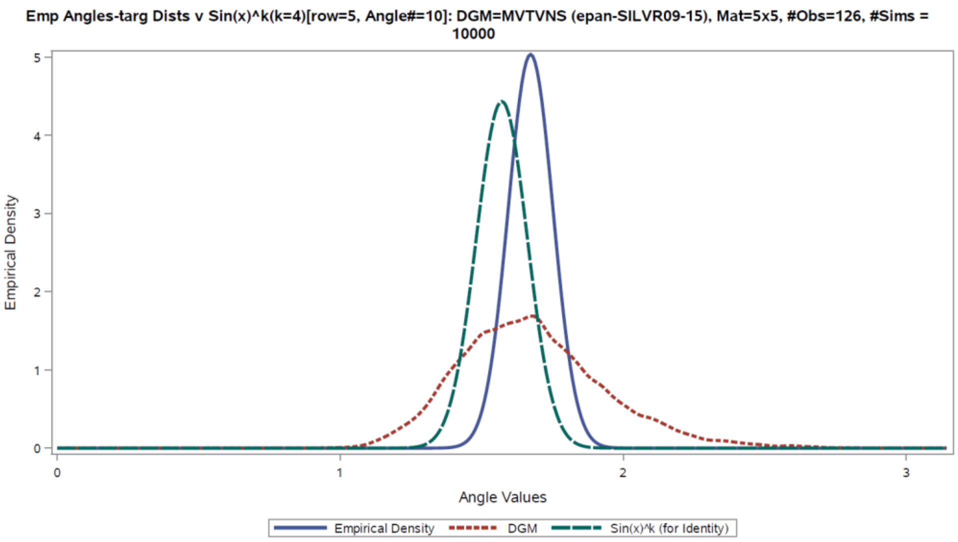
Angle Distributions – Targeted Scenario v Unconstrained

n = 6

n = 126



PRNG Seed = 12345
RunID = FTs5Mblk



PRNG Seed = 12345
RunID = FTb5Mblk

Correlation Matrices to CDF Matrices – Both Targeted Scenario

Correlation Matrices | **n=6** | **n=126**

1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.53	0.53	0.53	0.53	0.53		0.4874	0.4874	0.4874	0.4874	0.4874
0	0.1	0.2	-0.1	-0.1		0.47	0.47	0.47	0.47	0.47		0.5024	0.5024	0.5024	0.5024	0.5024
0	0.1	0.2	-0.1	-0.1		0.47	0.47	0.47	0.47	0.47		0.5326	0.5326	0.5326	0.5326	0.5326
0	0.1	0.2	-0.1	-0.1		0.46	0.46	0.46	0.46	0.46		0.5263	0.5263	0.5263	0.5263	0.5263
1	1	1	1	1												
0	0.1	0.2	-0.1	0.2		0.46	0.44	0.41	0.39	0.37		0.206	0.0938	0.042	0.0184	0.0068
0	0.1	0.2	-0.1	0.2		0.55	0.52	0.50	0.48	0.46		0.5734	0.4045	0.2671	0.169	0.0922
0	0.1	0.2	-0.1	0.2		0.55	0.55	0.55	0.55	0.55		0.5677	0.5677	0.5677	0.5677	0.5677
1	1	1	1	1												
0	0.1	0.2	-0.1	-0.1		0.67	0.65	0.64	0.63	0.72		0.9824	0.953	0.9189	0.8742	0.9996
0	0.1	0.2	-0.1	-0.1		0.63	0.61	0.60	0.59	0.69		0.9535	0.8954	0.8262	0.7569	0.9996
1	1	1	1	1												
0	0.1	0.2	-0.1	0.3		0.68	0.67	0.67	0.66	0.70		0.9978	0.9939	0.9902	0.986	0.9995
1	1	1	1	1												

CDF Matrices to Correlation Matrices – Both Targeted Scenario

CDF Matrices **n=6** **n=126**

						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.35	0.9		0.10	0.10	0.10	0.10	0.10		0.07	0.07	0.07	0.07	0.07
0.5	0.4	0.6	0.35	0.9		-0.11	-0.11	-0.11	-0.11	-0.11		-0.09	-0.09	-0.09	-0.09	-0.09
0.5	0.4	0.6	0.35	0.9		-0.11	-0.11	-0.11	-0.11	-0.11		-0.07	-0.07	-0.07	-0.07	-0.07
0.5	0.4	0.6	0.35	0.9		-0.15	-0.15	-0.15	-0.15	-0.15		-0.10	-0.10	-0.10	-0.10	-0.10
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.6	0.8		-0.03	0.25	-0.30	-0.30	-0.74		-0.05	-0.01	-0.09	-0.09	-0.18
0.5	0.4	0.6	0.6	0.8		0.22	0.47	-0.06	-0.06	-0.63		0.12	0.16	0.07	0.07	-0.04
0.5	0.4	0.6	0.6	0.8		0.01	0.01	0.01	0.01	0.01		0.04	0.04	0.04	0.04	0.04
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.4	0.3		0.55	0.75	0.32	0.72	0.91		0.37	0.40	0.34	0.40	0.44
0.5	0.4	0.6	0.4	0.3		0.49	0.67	0.19	0.65	0.54		0.32	0.35	0.28	0.35	0.37
						1	1	1	1	1		1	1	1	1	1
0.5	0.4	0.6	0.55	0.95		0.79	0.83	0.49	0.81	0.30		0.55	0.59	0.51	0.55	0.39
						1	1	1	1	1		1	1	1	1	1

CDF %Shift Matrices to Correlation Matrices – Both Targeted Scenario

CDF %Shift Matrices **n=6** **n=126**

					lnMatPr	-2.369	-2.101	-7.529	-6.449	-14.780		-2.292	-2.175	-7.204	-6.731	-12.865
					FNorm	0.889	0.927	1.620	1.913	1.884		0.122	0.117	0.318	0.325	0.530
					Rnk_lnMat	2	1	4	3	5		2	1	4	3	5
					Rnk_FNorr	1	2	3	5	4		2	1	3	4	5
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	60		0.10	0.10	0.10	0.10	0.10		0.07	0.07	0.07	0.07	0.07
20	-20	50	-50	60		-0.11	-0.11	-0.11	-0.11	-0.11		-0.09	-0.09	-0.09	-0.09	-0.09
20	-20	50	-50	60		-0.11	-0.11	-0.11	-0.11	-0.11		-0.07	-0.07	-0.07	-0.07	-0.07
20	-20	50	-50	60		-0.15	-0.15	-0.15	-0.15	-0.15		-0.10	-0.10	-0.10	-0.10	-0.10
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	70		0.25	-0.31	0.61	-0.66	0.78		-0.01	-0.09	0.06	-0.16	0.11
20	-20	50	-50	70		0.36	-0.19	0.71	-0.58	0.84		0.14	0.04	0.21	-0.03	0.28
20	-20	50	-50	70		0.01	0.01	0.01	0.01	0.01		0.04	0.04	0.04	0.04	0.04
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	80		0.65	0.14	0.92	0.14	0.98		0.41	0.34	0.46	0.29	0.54
20	-20	50	-50	80		0.57	0.02	0.67	-0.32	0.60		0.35	0.29	0.41	0.23	0.47
						1	1	1	1	1		1	1	1	1	1
20	-20	50	-50	90		0.81	0.16	0.70	-0.21	0.53		0.60	0.53	0.65	0.46	0.73
						1	1	1	1	1		1	1	1	1	1