The Challenges of, and Practical Solutions to, Capital Aggregation and Allocation Under Heavy-tailed, Empirical Loss Distributions

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I. What are Capital Aggregation & Capital Allocation?

- **We have a portfolio with *N* components, each with a (loss) distribution.**
- **We use various risk metrics to assess the riskiness and estimated the capital requirements of each component. Applying a risk metric to each standalone component (i.e. to each distribution) typically is straightforward:**

where X is the loss variable,

- **Value-at-Risk =** $VaR_{\alpha} = \inf\left\{x \in \mathbb{R} : F\left(x\right) \geq \alpha\right\}$, for $\alpha \in \left(0,1\right)$, where $F\left(x\right) = \text{cdf of } X$
- **Expected Shortfall =** $ES_{\alpha} = \left(1/\left[1-\alpha\right]\right)\int_{\alpha}^{1} VaR_{u}\,du$, for $\alpha \in \left(0,1\right)$
- **Expectile =** $e_{\alpha}\left(X\right) = \underset{x \in \mathbb{R}}{\arg\min}\left\{\alpha\left[E\left(\left(X-x\right)^{+}\right)\right]^{2} + \left(1-\alpha\right)\left[E\left(\left(X-x\right)^{-}\right)\right]^{2}\right\}$

- **<u>Capital Aggregation</u>: to apply the risk metric(s) to the entire portfolio, we must AGGREGATE the components of the portfolio and treat them as a multivariate loss distribution.**

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I. What are Capital Aggregation & Capital Allocation?

- **Capital Aggregation: to apply the risk metric(s) to the entire portfolio, we must AGGREGATE the components of the portfolio and treat them as a multivariate loss distribution.**

- **Under perfect dependence, losses across components occur perfectly in tandem, so for VaR for example, the 1-in-1000 year loss (α=0.999), on average, for all the components will occur in the same time period, and portfolio VaR equals their sum:**

$$\text{portfolio } VaR_\alpha = \sum_{i=1}^{N} VaR_\alpha^i$$

- **Under IMperfect dependence, losses across components DO NOT occur perfectly in tandem (i.e. the portfolio possesses some diversification), so:**

$$\text{portfolio } VaR_\alpha < \sum_{i=1}^{N} VaR_\alpha^i \quad \text{assuming subadditivity (see below)}$$

- **The size of the diversification benefit =** $\text{db} = \left[\sum_{i=1}^{N} VaR_\alpha^i - \text{portfolio } VaR_\alpha \right]$ **depends mainly on:**
  - **The degree to which dependence structure deviates from perfect dependence (e.g. if measured by correlation bounded by -1 and 1, the greater the deviation from positive 1, the greater the db, ceteris paribus).**
  - **The selected risk metric & its interaction with the marginal loss distributions**

I. What are Capital Aggregation & Capital Allocation?

- **Capital Aggregation is necessary for an accurate assessment of risk at the portfolio level.**
- **Capital Aggregation is desirable when diversification benefit is positive and material, as should often be the case in real world financial applications.***
- **Across most regulatory settings, capital aggregation largely has been acknowledged as appropriate, and has even been encouraged, for obtaining accurate assessments of portfolio-level risk.**

* In operational risk, for example, a widely cited range for typical diversification benefit is 25%-50% of total estimated capital (see Incisive Media, OR&R, 2009).

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I. What are Capital Aggregation & Capital Allocation?

- **Capital Allocation: Once capital has been aggregated, it must then be (re)allocated back to the components of the portfolio for business-decisioning and other strategic purposes.**
- **Note that the unarguable platinum standard for capital allocation is the Euler allocation principal, which has been shown to be either optimal or 'most appropriate' from a wide range of perspectives including the economic, game theoretic, portfolio optimization, insurance, and axiomatic (see Tasche, 2007).**
- **Conceptually, Euler allocation is the unique allocation solution for which the gradients of all components of the portfolio are equal. In other words, under Euler allocation, the marginal effect to portfolio-level risk of a dollar allocated to one component is equal to that of any other.**
  - **Euler Allocation principal: A function, ψ, that is differentiable and positively homogenous satisfies**

$$\psi = \sum_{i=1}^{N} w_i \frac{\partial \psi}{\partial w_i} \quad \text{where } w_i = \text{weights on portfolio components.}$$

  - **Note that while standalone VaR is not differentiable, it is positively homogeneous with** $f(\alpha x) = \alpha^n f(x)$ where $n =$ degree of homogeneity; $\alpha > 0$.

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I. What are Capital Aggregation & Capital Allocation?

- Capital Aggregation: Methods for aggregation include
  - Copulas (see Luo and Shevchenko, 2011)
  - Factor Models (see Embrechts et al., 1999a)
  - Factor Copulas (see Oh and Patten, 2015)
  - Other (see Arakelian and Karlis 2014; Bernard and Vanduffel 2014; Dacoragna et al., 2016; Dhaene et al., 2013; Polanski et al 2013; and Reshetar, 2008)

- Of course, factor models can be the platinum standard when extant data and methodological implementation permit the estimation of fully specified econometric models that adequately incorporate the dependence structure of multiple components of the portfolio. Unfortunately this is often not the case, especially when aggregating at the enterprise level.

I. What are Capital Aggregation & Capital Allocation?

- **We focus herein on copulas.**
- **Copulas are the most widely used method of capital aggregation in part because they often have fewer data and methodological requirements compared to frameworks relying (solely) on factor models.**
- **Their widespread usage also is due to their tractability and an extremely convenient and useful characteristic of their estimation and implementation: the dependence structure and the marginal loss distributions are completely independently specified and/or estimated (see Sklar, 1959; Meucci, 2011).**
- **In other words, the dependence structure is in no way related to the marginal loss distributions, and each can be specified and/or estimated separately.**

I. What are Capital Aggregation & Capital Allocation?

- **We note here that one of the most widely used copulas, the meta t-copula,* has one major limitation that is particularly relevant under HEMLD: it assumes identical tail dependence across all components of the portfolio, which is generally implausible (see Wagner and Wenger, 2009), especially under HEMLD for enterprise-level dependence structures.**
- **So for practical usage under HEMLD, we note that the varying df meta t-copula of Luo and Shevchenko (2011) circumvents this limitation by allowing the parameter measuring tail dependence – the degrees of freedom (df) – to vary for each component of the portfolio. This copula nests the standard meta t-copula (and so its estimation will always be at least as good as the standard case), and it is much more straightforward to implement and estimate than nested t-copulas (see Wagner and Wenger, 2009). The copula is presented below.**

* Technically , a "t-copula" whose marginals are not distributed as the student's t distribution is a meta t-copula. This almost always is the case in practice, although the copula is widely referred to soley as a "t-copula," dropping the 'meta' prefix.

I. What are Capital Aggregation & Capital Allocation?

Per Luo and Shevchenko (2011):

The meta t-copula, $\tilde{t}_v$, with multiple degrees of freedom, $v=\left(v_1,\ldots,v_n\right)'$, has the following explicit integral expression:

$$C_v^{\Sigma}(\mathrm{u})=\int_0^1 \Phi_{\Sigma}\left(z_1(u_1,s),\ldots,z_n(u_n,s)\right)ds \text{ , with density}$$

$$c_v^{\Sigma}(\mathrm{u})=\frac{\partial^n C_v^{\Sigma}(\mathrm{u})}{\partial u_1\ldots\partial u_n}\int_0^1 \phi_{\Sigma}\left(z_1(u_1,s),\ldots,z_n(u_n,s)\right)\prod_{k=1}^{n}\left[w_k(s)\right]^{-1}ds\left(\prod_{k=1}^{n} f_{v_k}(x_k)\right)^{-1}$$

where

$z_k(u_k,s)=t_{v_k}^{-1}(u_k)/w_k(s)$, $k=1,2,\ldots,n$ where $t_{v_k}^{-1}(\ )$ is the inverse of the student's t distribution with v_k df

$w_k(s)=G_{v_k}^{-1}(s)=\sqrt{v_k/\chi_{v_k}^{-1}(s)}$ where $\chi_{v_k}^{-1}(s)$ is the inverse of the cdf of the Chi-square distribution with v_k df

$s=$ one random draw from the uniform $(0,1)$ distribution

$\phi_{\Sigma}(z_1,\ldots,z_n)=\exp\left(-\frac{1}{2}\mathrm{z}'\Sigma^{-1}\mathrm{z}\right)\Big/\left[(2\pi)^{n/2}\sqrt{\det\Sigma}\right]$ is the multivariate standard Normal density

$x_k=t_{v_k}^{-1}(u_k)$, $k=1,2,\ldots,n$

$f_v(x)=\left(1+x^2/v\right)^{-\frac{v+1}{2}}\Gamma\left(\frac{1}{2}(v+1)\right)\Big/\left[\Gamma\left(\frac{v}{2}\right)\sqrt{v\pi}\right]$ is the univariate student's t density*

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*** Note the preprint contains a typo in this formula.**

I. What are Capital Aggregation & Capital Allocation?

- **Capital Allocation: Common methods for allocation include**
  - **Euler allocation principal (see Tasche, 2007; Tasche, 2009; and Holden, 2008)**
  - **Shapley method (see Shapley, 1953; Balog, 2010)**
  - **Haircut principal / Activity based method (see Hamlen et al., 1977)**
  - **Beta method / Covariance principal (see Panjer, 2002)**
  - **Incremental method (see Jorion, 2007)**
  - **Cost gap method (see Driessen and Tijs, 1985)**
- **The above are not always mutually exclusive (e.g. a Shapely implementation, although often very cumbersome, can be consistent with the Euler principal), and highly dependent upon the risk metric used and HOW they are used.**
- **For example, Euler consistent allocation requires risk metric that is A. positively homogenenous, and B. differentiable. Standalone VaR satisfies condition A. but does not satisfy condition B. However, when a kernel is applied to VaR (see Tasche, 2009), B. is satisfied approximately.**

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II. Common Real World Constraints: HEMLD

- **When aggregating capital at high levels, say, across an institution's risk types (e.g. operational, credit, and market), it is not uncommon for marginal loss distributions (themselves often the result of aggregation) to be generated by non-trivial simulations, and/or simulations of simulations, which typically will have no parametric closed form (see Wagner and Wenger, 2009).**
- **Even in the absence of simulation-based loss distributions, the lack of closed-form distributions often is by design and/or (implicit) regulatory requirement. For example, AMA-consistent Operational Risk loss distributions typically have no closed form parametric representation under the Loss Distribution Approach (which is used, and 'regulator approved,' almost ubiquitously). Consequently, these are 'empirical' loss distributions, and they also typically are very heavy-tailed (see Opdyke, 2014). While not as heavy-tailed as Operational risk loss distributions in most cases, Credit risk distributions used at the enterprise (portfolio) level certainly tend not to be light-tailed, and often aren't far behind their OpRisk counterparts in terms of the heaviness of their tails.**

II. Common Real World Constraints: HEMLD

- **So when aggregating capital at the top-of-the-house, where the diversification benefit is often largest in absolute terms, we often have Heavy-tailed, Empirical (i.e. nonparametric) Marginal Loss Distributions (HEMLD). Both of these characteristics pose computational challenges when aggregating (and allocating) capital.**
- **Take the most commonly used aggregation method: copulas.**
- **Under HEMLD, the heavy-tailedness and empirical nature of the marginal loss distributions interact to increase computational demands of a copula simulation* by many orders of magnitude.**
- **Due to the fact that the distributions are heavy-tailed, we need larger empirical loss distributions (i.e. many more simulated observations), to adequately represent the tails of the distributions. And of course, by definition, the tails cannot be extrapolated via parametric estimation.****
- **So we have very large empirical loss distributions (e.g. 10 million observations) that when translated into large empirical cumulative loss distributions (cdf's), still are very large (e.g. 1 million ±), and these need to be inverted post-copula-simulation to obtain the multivariate loss distribution.**

* In practice, almost all copula implementations are based on simulations: very few copulae can be implemented with fully analytical representations. See Dacoragna et al. (2016) for a recent exception.

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II. Common Real World Constraints: HEMLD

- **Inverting the empirical cdf's is straightforward conceptually – it just requires a merge / join of the empirical cdf and the copula. But because of the heavy-tailed nature of the marginal loss distributions, the copula ALSO must generate large numbers of simulations (e.g. 10 million; the industry standard of 1 million is not sufficient) to adequately represent the extreme tails, let alone capture any extant tail dependence. So this becomes a merge / join of two very large datasets.**
- **These issues remain even when more involved sampling schemes that over-represent the distributional tails (e.g. Importance Sampling) can be employed.**
- **Merge / Join: even when done as efficiently as possible, takes way too long (hours/days on a desktop).**
- **Hashing: typically is much faster, but due to the SIZE of both the empirical cdf's and simulated copula, memory requirements are too great, and on all but the largest computers, it crashes.**

** 'Curve fitting' to approximate the empirical loss distribution has never been acceptable practice in Operational Risk from a regulatory perspective, and many would argue – whether or not in a regulatory capacity – that this exercise would be characterized by very high estimation variance in the tails of the (approximated) loss distributions, which would defeat the purpose of the approximation. In short, attempting to impose a parametric form creates an additional and unnecessary layer of estimation error.

II. Common Real World Constraints: HEMLD

- CATA: an alternate approach, based on Opdyke (2011 and 2013), avoids both obstacles: it is not only often orders of magnitude faster than merging / joining, but also handles memory much more efficiently than hashing and does not crash on cdf's and copulas more than an order of magnitude larger than those that hashing can handle.
- CATA also is so fast (minutes vs. hours on a desktop) that it avoids the need for more involved sampling schemes (like Importance Sampling), which, even when useable, add an additional layer of distributional approximation.
- Convergence Algorithm using Temporary Arrays (CATA) is stable, robust, and always finds a solution. It converts the empirical cdf into a temporary array per Opdyke (2011 and 2013), thus converting a column of data into a row of data which in SAS® resides entirely in very efficiently used memory. Then bi-section or a variant* (see Galdino, 2011) is applied to lookup the cdf value that corresponds to each copula value that is simulated one value at a time. Once the cdf is inverted, the value is again saved as a field (column) of data.
- Even when the empirical cdf is over 1 million values, the bi-section typically converges in a dozen or so iterations with accuracy to 1.0e-15 (the highest a desktop computer chip can achieve).

* Variants that exploit the approximate shapes of the empirical cdf's (they are very concave) can yield additional algorithmic efficiencies here.

III. Re-examining VaR vs. ES under HEMLD

- As mentioned above, the risk metric used has a very strong effect on capital allocation results.
- The most commonly used risk metric for standalone capital estimation and aggregation is VaR, with ES also being widely used; for capital allocation, both ES and VaR are widely used, with the former arguably used more widely.
- ES has nice theoretical properties, mainly that it is a ‘coherent’ risk measure per Artzner et al. (1999), where ‘coherent’ is defined as satisfying the four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity.
- It is worth emphasizing that only its lack of global subadditivity prevents VaR from being classified as a ‘coherent’ risk measure.
- Also note that while seminal, the axioms of coherence are not inviolate: some have proposed modification or replacement of some of these axioms under various real-world conditions, especially in the regulatory realm (see, for example, Dhaene et al., 2008; and Föllmer and Weber, 2015, for definitions of alternate sets of axioms defining ‘convex’ and ‘monetary’ risk measures).

III. Re-examining VaR vs. ES under HEMLD

- Unfortunately, theoretical properties of risk metrics, in this case ES, are not always consistent with the empirical results of capital allocations based on real-world loss datasets of large financial institutions (i.e. regulated ones): this is especially true under HEMLD.
- It has been well documented in the literature, both analytically and empirically, that ES is associated with much greater estimation variance than is VaR, ceteris paribus (i.e. using the same percentile values for α; for example, see Yamai and Yoshiba, 2002).
- Under HEMLD, even when ES is finite, this variance can be extremely large, and lead to superadditivity EMPIRICALLY, and this result can be reasonably stable and consistent (i.e. not simply the result of a few extreme simulations out of many millions).

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III. Re-examining VaR vs. ES under HEMLD

- **Objections to VaR's use in capital aggregation/allocation are twofold:**
  1. **not globally subadditive, and**
  2. **not differentiable, which prevents allocation consistent with the Euler allocation principal.**

Per 1.:

- **While VaR is not globally subadditive, it has been shown to be subadditive, under many conditions, in the relevant range of the loss distribution, i.e. far into the tails (see Cumperayot et al., 2000; Danielsson et al., 2005; Danielsson et al., 2013; Embrechts et al., 2008; and Klueppelberg and Resnick, 2008).**
- **VaR appears to be empirically subadditive under HEMLD for large alpha (i.e. out in the tails), which is the only place aggregation and allocation are done (see Wagner and Wenger, 2009).**
- **Finally, the most widely cited examples of VaR's superadditivity are recognized as not based on real-world loss data from large financial institutions (e.g. Gaussian distributions; see Embrechts et al., 1999b), and even dismissed as 'ad hoc' (see Wagner and Wenger, 2009).**

III. Re-examining VaR vs. ES under HEMLD

- **So this is arguably somewhat paradoxical: ES is globally subadditive in theory, but under HEMLD its empirical results can violate this axiom and exhibit superadditivity. In contrast, VaR is NOT globally subadditive, but under HEMLD, empirical results render this axiom moot: VaR appears to be subadditive empirically far into the tails, where it matters.**
- **For some, however, this may seem less paradoxical than merely very intriguing: perhaps this group would include practitioners more familiar with provocative divergences between purely theoretical results and challenging, empirical realities.**

Per 2.:

- **While (standalone) VaR is not differentiable, Tasche (2009) applies a Nadayara-Watson kernel to it to make it approximately so.**

$$VaR_{\alpha}\left(X_i \mid X\right) \approx \frac{\sum_{k=1}^{T} x_{i.k}\,\varphi\left(\left[VaR_{\alpha}\left(\hat{X}+b\varepsilon\right)\right]\big/ b\right)}{\sum_{k=1}^{T} \varphi\left(\left[VaR_{\alpha}\left(\hat{X}+b\varepsilon\right)\right]\big/ b\right)}, \quad \text{where } x_k = \sum_{k=1}^{N} x_{i.k} = \text{ sum across } N \text{ portfolio components; } T = \#\text{ of simulated time periods;}$$

X and X_i are the respective random (loss) variables, $\hat{X}$ is the empirical realization of the former;

$\alpha = 0.999$; ε is a standard normal random variable and φ is the standard normal kernel;

b is the kernel bandwidth (e.g. $b = 1.06\hat{\sigma}T^{-1/5}$ where $\hat{\sigma}$ is standard deviation (estimated based on $\hat{X}$))

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III. Re-examining VaR vs. ES under HEMLD

Per 2.:

- **While (standalone) VaR is not differentiable, Tasche (2009) applies a Nadarya-Watson kernel to it to make it approximately so.**
- **This allows for allocation approximately consistent with the Euler allocation principle, and eliminates the remaining objection to VaR's use.**
- **Finally, VaR is increasingly being cited in the relevant aggregation and allocation literatures as being <u>empirically</u> subadditive over the relevant ranges of α, and that this result is a robust one (see Wagner and Wenger, 2009)***

* p.136 "… In all calculations in these papers, there arises a diversification benefit in the process of aggregating via VaR. Hence subadditivity is not violated in the numerical examples arising in these works. Many practitioners have confirmed these conclusions in various other settings and parametrizations. Going beyond mere numerical evidence, Danielsson, Jorgensen, Sarma, and deVries (2005) provide a class of examples where subadditivity can indeed be proved, both for independent and for dependent risks. …Embrechts, Neshlehova, and Wűthrich (2008) and Klűppelberg and Resnick (2008) prove subadditivity in a large class of models typical for OpR and in fact even characterize some classes of models by the subadditivity property."

III. Re-examining VaR vs. ES under HEMLD

Per 2.:

Additional Advantages of VaR over ES under HEMLD:

- **Using a single risk metric for both aggregation and allocation avoids potential misunderstanding, misinterpretation, and internal inconsistencies in translations between multiple risk metrics (e.g. using VaR with one value of α for aggregation, and ES with another value of α that yields the same VaR with the original α, is not uncommon). Using a single risk metric also greatly simplifies and makes consistent the cascading of allocations from higher levels (e.g. across risk types at the enterprise level) to lower levels (e.g. business units).**
- **Under HEMLD, in addition to growing evidence shows that VaR is (empirically) coherent, while ES, at least in some cases, is not, VaR also is more stable, ceteris paribus (for example, see Yamai and Yoshiba, 2002). Consequently, under HEMLD, VaR maintains notable advantages over ES, and we remain unaware of any real advantages of ES over VaR.***

* One possible exception to this are issues related to ease of decomposition, but these may be rendered irrelevant anyway under HEMLD, that is, without analytically convenient parametric marginal loss distributions.

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IV. Summary and Conclusions

- **Capital Aggregation and Allocation are necessary and desirable for the accurate assessment of portfolio level risk and obtaining diversification benefit, which at the enterprise level can be substantial ($ billions).**
- **Heavy-tailed, Empirical Marginal Loss Distributions (HEMLD), commonly encountered when aggregating/allocating capital at the enterprise level, pose non-trivial empirical and computational challenges to this exercise.**
- **Presented herein is a stable, efficient algorithm, based on widely used and proven methods, that circumvents these challenges posed by HEMLD.**
- **HEMLD also prompts a re-examination of the selection of risk metrics for capital aggregation and allocation.**
- **Under HEMLD, VaR is empirically subadditive and coherent, while ES, at least in some cases, is not. VaR also is more stable, ceteris paribus. We remain unaware of any advantages of ES over VaR under HEMLD.**
- **Additionally, applying a kernel to VaR a la Tasche (2009) renders capital allocation based on it approximately consistent with the Euler allocation principal, thus removing the only other objection to the use of VaR over ES for either capital aggregation or allocation.**

IV. Summary and Conclusions

“Measurement is the first step that leads to control and eventually to improvement. If you can’t measure something, you can’t understand it. If you can’t understand it, you can’t control it. If you can’t control it, you can’t improve it.”

- **H.J. Harrington**

Consistent with Harrington’s adage above, we have presented herein a very efficient algorithm to utilize a fully flexible copula when faced with Heavy-tailed Empirical Marginal Loss Distributions (HEMLD). Under these data conditions, which are commonly encountered at the enterprise level, VaR is empirically coherent, while ES sometimes is not, and VaR is more stable, ceteris paribus. VaR-based allocation also can be consistent with the Euler principal. The often very large diversification benefit generated by capital aggregation makes its accurate and stable measurement, let alone its optimal allocation, of paramount importance.

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